THE PROBABILITY METHOD
FOR UPPER BOUNDS IN
DOMINATION THEORY

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Abstract

Domination is a rapidly developing area of research in graph theory, and its various applications to ad hoc networks, distributed computing, social networks and web graphs partly explain the increased interest. This thesis focuses on domination theory, and the main aim of the study is to apply a probabilistic approach to obtain new upper bounds for various domination parameters.

Chapters 2 and 3 are devoted to $k$-domination, $k$-tuple domination, $k$-total domination, $\alpha$-domination and $\alpha$-rate domination in graphs. A review of well-known results is given, and the new results are presented. These new upper bounds generalize two classical bounds for the single domination number and also improve a number of known bounds for the $k$-domination and $k$-tuple domination numbers. Effective randomized algorithms are given for finding $k$-dominating, $k$-tuple dominating, $\alpha$-dominating and $\alpha$-rate dominating sets, whose expected sizes satisfy the above upper bounds. These algorithms follow from the probabilistic constructions used to prove the corresponding upper bounds.

Similar research is carried out for the the global and Roman domination parameters in Chapter 4. New upper bounds for the global domination and Roman domination numbers are presented, and it is proved that these results are asymptotically best possible. Moreover, the upper bounds for the restrained domination and total restrained domination numbers for large classes of graphs are given, and it is shown that, for almost all graphs, the restrained domination number is equal to the domination number, and the total restrained domination number is equal to the total domination number.

Signed domination is another domination parameter studied in Chapter 5. This concept is closely related to combinatorial discrepancy theory. New upper and lower bounds for the signed domination number are presented. These new bounds improve a number of known results. Moreover, Füredi–Mubayi’s conjecture is rectified.
To my loving parents
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Chapter 1

Introduction

1.1 Background

Graph-theoretical ideas date back to at least the 1730’s, when Leonhard Euler published his paper on the problem of Seven Bridges of Königsberg [12]. This puzzle asks whether there is a continuous walk that crosses each of the seven bridges of Königsberg only once and if so, whether a closed walk can be found. Furthermore, the large part of graph theory has been motivated by the study of games and recreational mathematics. Graphs are very convenient tools for representing the relationships among objects, which are represented by vertices. In their turn, relationships among vertices are represented by connections. In general, any mathematical object involving points and connections among them can be called a graph or a hypergraph. For a great diversity of problems such pictorial representations may lead to a solution. Examples of such applications include databases, physical networks, organic molecules, map colorings, signal-flow graphs, web graphs, tracing mazes as well as less tangible interactions occurring in social networks, ecosystems and in a flow of a computer program. The graph
models can be further classified into different categories. For instance, two atoms in an organic molecule may have multiple connections between them, an electronic circuit may use a model in which each edge represents a direction, or a computer program may consist of loop structures. Therefore, for these examples we need multigraphs, directed graphs or graphs that allow loops. Thus, graphs can serve as mathematical models to solve an appropriate graph-theoretic problem, and then interpret the solution in terms of the original problem.

At present, graph theory is a dynamic field in both theory and applications. Graphs can be used as a modelling tool for many problems of practical importance. For instance, a network of cities, which are represented by vertices, and connections among them make a weighted graph. The well-known travelling salesman problem asks for the shortest possible tour, which visits all the cities exactly once. And there are numerous applications like this.

Scientific conjectures are another important research area of graph theory. Scientific conjectures, obviously, express the most interesting theoretic statements, which are neither proved nor disproved. They are basically posed in order to draw attention of scientific community and advance progress in the corresponding field. Some conjectures have had a fundamental impact. For example, the development of graph theory over the last four decades has been strongly influenced by the Strong Perfect Graph Conjecture and perfect graphs introduced by Berge in the early 1960s [9, 10]. This famous conjecture has been open for about 40 years, and various attempts to prove it have given rise to many powerful methods, important concepts and interesting results in graph theory. The conjecture was recently proved in [21], and now it is referred to as the Strong Perfect Graph Theorem.
1.2 Domination

In this section, the concept of domination in graphs and its various applications for real life problems are discussed. Moreover, a review of existing results on domination is presented. The terminology and notations used throughout this thesis are also given.

1.2.1 Preliminaries and notion

Domination theory, in particular various domination parameters in graphs are the main object of study in this thesis. The basic familiarity with the notion of graphs, the concept of domination and standard algorithmic tools is assumed. Some fundamental definitions and notations used throughout this thesis are presented in this subsection. Some others will be given later when necessary. For an overview on the general theory of graphs, the reader can refer to Harary’s *Graph Theory* \[43\] and Berge’s *Graphs and Hypergraphs* \[11\].

A graph or undirected graph $G$ is an ordered pair $G = (V, E)$ where $V$ is a set, elements of which are called vertices or nodes, and $E$ is a set of unordered pairs of distinct vertices called edges or lines. A finite graph is a graph such that $V(G)$ and $E(G)$ are finite sets. The *complement* of a graph $G$ is the graph $\overline{G}$ with the same vertex set such that two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in $G$. All graphs studied in this thesis are finite and undirected without loops and multiple edges. In a small number of cases, when dealing with generalizations of concepts for graphs, we also refer to hypergraphs. A *hypergraph* is a pair $\mathcal{H} = (V, \mathcal{E})$, where $V$ is the vertex set and $\mathcal{E} = \{E_1, ..., E_m\}$, a family of subsets of $V$, is the hyperedge set.
If $G$ is a graph of order $n$, then $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices in $G$, $d_i$ denotes the degree of $v_i$ and $d = \sum_{i=1}^{n} d_i/n$ is the average degree of $G$. A vertex of degree 0 is an isolated vertex. In a graph $G$, a walk from vertex $v_0$ to vertex $v_n$ is an alternating sequence $W = \langle v_0, e_1, v_1, e_2, ..., v_{n-1}, e_n, v_n \rangle$ of vertices and edges such that the vertices $v_{i-1}$ and $v_i$ are the endpoints of the edge $e_i$, for $i = 1, ..., n$. In a graph $G$, the distance from vertex $u$ to vertex $v$, denoted by $d(u,v)$ is the length of the shortest walk from $u$ to $v$, i.e. the number of edges in such a walk. If there is no walk from $u$ to $v$, then $d(u,v) = \infty$. For $v \in V$ and $S \subseteq V$, the distance from $v$ to $S$ is defined as $d(v,S) = \min_{s \in S} \{d(v,s)\}$. A path is a walk with no repeated vertices. A graph $G$ is said to be connected if there exists a path between every pair of vertices and disconnected otherwise. A graph $G$ is called $k$-connected if removal of $k$ or more vertices makes the graph disconnected. A graph $G$ is said to be of size $m$ if it has $m$ edges. Let $N(x)$ denote the neighbourhood of a vertex $x$. Also, let $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of vertices of $G$, respectively. Put $\delta = \delta(G)$ and $\Delta = \Delta(G)$.

A set $D$ is called a dominating set if every vertex not in $D$ is adjacent to one or more vertices in $X$. For example, two dominating sets of a sample graph are illustrated in Figure 1.1.

Note that there are several different equivalent definitions of a dominating set and each of them provides a different aspect of the concept of domination. Some of them, provided in [48], are listed below. A set $D \subseteq V$ is a dominating set if and only if one of the following is true.

- **Vertex Set Covering**: each vertex $v \in V - D$ has at least one adjacent vertex $u \in D$ (i.e., is covered by a vertex).
1.2 Domination

- **Set intersection**: for every vertex \( v \in V \), \(|N[v] \cap D| \geq 1\).

- **Distance from the set**: for every vertex \( v \in V - D \), the distance \( d(v, D) = 1 \).

- **Union of neighbourhoods**: \( N[D] = V \).

A *minimal dominating set* in a graph \( G \) is a dominating set that contains no dominating set as a proper subset. The minimum cardinality of a dominating set of \( G \) is the *domination number* \( \gamma(G) \). The domination number can also be defined in terms of a dominating function as shown in [47]. Let \( f \) be a function \( f : V \rightarrow \{0, 1\} \). The function \( f \) is called a *dominating function* if for every vertex \( v \in V \),

\[
  f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1.
\]

A dominating function \( f \) is a *minimal dominating function* if there does not exist a dominating function \( g \neq f \), for which \( g(v) \leq f(v) \) for every \( v \in V \). The *weight* of a function \( f \), denoted by \( w(f) \), is equal to \( \sum_{v \in V} f(v) \). Then the domination number can be defined as

\[
  \gamma(G) = \min_{f \in \mathcal{D}} \{w(f)\},
\]

where \( f \) is a dominating function on \( G \).
In general, many domination parameters can be defined by combining domination with another graph theoretical property. Harary and Haynes [44] formalized this concept with the following definition by imposing an additional constraint on the dominating set: For a given property \( P \), the \textit{conditional domination number} \( \gamma(G:P) \) is the smallest cardinality of a dominating set \( D \subseteq V \) such that the induced subgraph \( G[D] \) satisfies property \( P \). Thus, by considering different properties \( P \), many new invariants of domination can be defined. An alternative approach is to impose an additional requirement on the set of edges between \( D \) and \( V - D \). For example, for \( k \)-domination it is required that each vertex outside the dominating set has at least \( k \) neighbours in the dominating set.

Since \( n \) is finite, the number of dominating sets of \( G \) with minimum cardinality is finite too. It is easy to see that, for a given graph \( G \), the domination number can have a value from the following range: \( 1 \leq \gamma(G) \leq n \). In particular, \( \gamma(G) = 1 \) if and only if \( \Delta = n - 1 \), and the equality for the upper bound is true if and only if \( \Delta = 0 \). The question whether there exists a polynomial algorithm for determining \( \gamma(G) \) naturally arises. However, for arbitrary graphs, there is no algorithm that has better time complexity than algorithms with exponential time complexity. Garey and Johnson [40] have shown that domination problem is NP-complete for arbitrary graphs. Thus, it is of importance to determine bounds for \( \gamma(G) \) and various similar parameters. This thesis focuses on finding new upper bounds for multiple and other domination parameters using a probabilistic approach.
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1.2.2 Applications

Domination is a rapidly developing area of research in graph theory. The concept of domination has existed for a long time and early discussions on the topic can be found in works of Ore [69] and Berge [11]. The summary of the literature shows the following wide-known problems, which are considered among the earliest applications for dominating sets.

Queens Problem

This problem was mentioned by Ore in [69]. According to the rules of chess a queen can, in one move, advance any number of squares horizontally, diagonally, or vertically (assuming that no other chess figure is on its way). How to place a minimum number of queens on a chessboard so that each square is controlled by at least one queen? See one of the solutions in Figure 1.2.

Figure 1.2: Queens dominating the chessboard
1.2 Domination

Locating Radar Stations Problem

The problem was discussed by Berge in [11]. A number of strategic locations are to be kept under surveillance. The goal is to locate a radar for the surveillance at as few of these locations as possible. How a set of locations in which the radar stations are to be placed can be determined?

Problem of Communications in a Network

Suppose that there is a network of cities with communication links. How to set up transmitting stations at some of the cities so that every city can receive a message from at least one of the transmitting stations? This problem was discussed in detail by Liu in [60].

Nuclear Power Plants Problem

A similar known problem is a nuclear power plants problem. There are various locations and an arc can be drawn from location \( x \) to location \( y \) if it is possible for a watchman stationed at \( x \) to observe a warning light located at \( y \). How many guards are needed to observe all of the warning lights, and where should they be located?

At present, domination is considered to be one of the fundamental concepts in graph theory and its various applications to ad hoc networks, biological networks, distributed computing, social networks and web graphs [11, 25, 27, 47] partly explain the increased interest. Such applications usually aim to select a subset of nodes that will provide some definite service such that every node in the network is ‘close’ to some node in the subset. The following examples show when the
concept of domination can be applied in modelling real-life problems.

Modelling Biological Networks

Using graph theory as a modelling tool in biological networks allows the utilization of the most graphical invariants in such a way that it is possible to identify secondary RNA (Ribonucleic acid) motifs numerically. Those graphical invariants are variations of the domination number of a graph. The results of the research carried out in [49] show that the variations of the domination number can be used for correctly distinguishing among the trees that represent native structures and those that are not likely candidates to represent RNA.

Modelling Social Networks

Dominating sets can be used in modelling social networks and studying the dynamics of relations among numerous individuals in different domains. A social network is a social structure made of individuals (or groups of individuals), which are connected by one or more specific types of interdependency. The choice of initial sets of target individuals is an important problem in the theory of social networks. In the work of Kelleher and Cozzens [55], social networks are modelled in terms of graph theory and it was shown that some of these sets can be found by using the properties of dominating sets in graphs.

Facility Location Problems

The dominating sets in graphs are natural models for facility location problems in operational research. Facility location problems are concerned with the location of one or more facilities in a way that optimizes a certain objective such
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as minimizing transportation cost, providing equitable service to customers and capturing the largest market share.

Coding Theory

The concept of domination is also applied in coding theory as discussed by Kalbfleisch, Stanton and Horton [54] and Cockayne and Hedetniemi [24]. If one defines a graph, the vertices of which are the $n$-dimensional vectors with coordinates chosen from $\{1, \ldots, p\}$, $p > 1$, and two vertices are adjacent if they differ in one coordinate, then the sets of vectors which are $(n, p)$-covering sets, single error correcting codes, or perfect covering sets are all dominating sets of the graph with determined additional properties.

Multiple Domination Problems

An important role is played by multiple domination. Multiple domination can be used to construct hierarchical overlay networks in peer-to-peer applications for more efficient index searching. The hierarchical overlay networks usually serve as distributed databases for index searching, e.g. in modern file sharing and instant messaging computer network applications. Dominating sets of several kinds are used for balancing efficiency and fault tolerance [27] as well as in the distributed construction of minimum spanning trees. Another good example of direct, important and quickly developing applications of multiple domination in modern computer networks is a wireless sensor network. A wireless sensor network (WSN) usually consists of up to several hundred small autonomous devices to measure some physical parameters. Each device contains a processing unit and a limited memory as well as a radio transmitter and a receiver to be able
to communicate with its neighbours. Also, it contains a limited power battery and is constrained in energy consumption. There is a base station, which is a special sensor node used as a sink to collect information gathered by other sensor nodes and to provide a connection between the WSN and a usual network. A routing algorithm allows the sensor nodes to self-organize into a WSN. As stated in [66], an important goal in WSN design is to maximize the functional lifetime of a sensor network by using energy efficient distributed algorithms, networking and routing techniques. To maximize the functional lifetime, it is important to select some sensor nodes to behave as a backbone set to support routing communications. The backbone set can be considered as a dominating set in the corresponding graph. Dominating sets of several different kinds have proved to be useful and effective for modelling backbone sets. In the recent literature (e.g., see [27]), particular attention has been paid to construction of $k$-connected $k$-dominating sets in WSNs, and several probabilistic and deterministic approaches have been proposed and analyzed. The backbone set of sensor nodes should be selected as small as possible and, on the other hand, it should guarantee high efficiency and reliability of networking and communications. This trade-off requires construction of multiple dominating sets providing energy efficient and reliable data dissemination and communication protocols.

A homogeneous WSN consists of wireless sensor devices of the same kind. All the devices have the same set of limited resources and, originally, no hierarchy is imposed on the network structure and communications. In a network of this kind, the only special sensor node is a base station. For all the other nodes, it is necessary to construct and switch the backbone sets and communications efficiently so that all the network nodes stay in operation as long as possible. Therefore,
in this case, it is important to be able to construct and switch dominating sets and route communications uniformly and efficiently with respect to the energy consumption of each particular sensor node. This has to be done to optimize the functional lifetime of the whole network.

Usually, a WSN is mathematically modelled as a unit or quasi-unit disk graph. These are the most natural and general graph models for a WSN. In a unit disk graph model, nodes correspond to sensor locations in the Euclidean plane and are assumed to have identical (unit) transmission ranges. An edge between two nodes means that they can communicate directly, i.e. the distance between them is at most one. A survey of known results on unit disk graphs, including algorithms for constructing dominating sets, can be found in [59]. A quasi-unit disk graph model [57] takes into consideration possible transmission obstacles and is much closer to reality: we are sure to have an edge between two nodes if the distance between them is at most a parameter $d$, $0 < d < 1$. If the distance between two nodes is in the range from $d$ to 1, the existence of an edge is not specified. A description of several more restricted geometric graph models for WSN design, e.g. the related neighbourhood graph, Gabriel graph, Yao graph etc., can be found in [59].

Domination is an area in graph theory with an extensive research activity. A book by Haynes, Hedetniemi and Slater [47] on domination published in 1998 lists 1222 articles in this area.

1.2.3 Review of Existing Results

This section surveys well-known general results on domination. The results presented here are mainly devoted to upper bounds of the domination number. Since
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this research involves multiple domination parameters, the review of results for other domination parameters has been carried out as well, and it is presented in the corresponding chapters.

The following theorems on dominating sets in graphs are the first results about domination and were presented by Ore in his book *Theory of Graphs* [69].

**Theorem 1 ([69])** If \( G \) is a graph with no isolated vertices, then the complement \( V - D \) of every minimal dominating set \( D \) is a dominating set.

**Theorem 2 ([69])** A dominating set \( D \) is a minimal dominating set if and only if for each vertex \( u \in D \), one of the following two conditions holds: \( u \) is an isolated vertex in \( D \); there exists a vertex \( v \in V - D \) for which \( N(v) \cap D = \{u\} \).

**Theorem 3 ([69])** Every connected graph \( G \) of order \( n \geq 2 \) has a dominating set \( D \) whose complement \( V - D \) is also a dominating set.

Further, the results on the domination number will be classified in categories.

**Bounds in Terms of Order and Size**

A classical result of Vizing [84] relates the size \( m \) and the domination number \( \gamma(G) \) of a graph \( G \) of order \( n \):

**Theorem 4 ([84])** For any graph \( G \),

\[
m \leq \left\lfloor (n - \gamma(G))(n - \gamma(G) + 2)/2 \right\rfloor.
\]

The equality in the bound of Theorem 4 can be achieved by considering a complete graph with \( n - \gamma(G) + 2 \) vertices, removing a minimum edge cover and adding
γ − 2 isolated vertices. In [77] Sanchis obtained similar results for connected graphs, and recently Fisher, Fraughnaugh and Seager [35] derived a tight result, which can be formulated as follows:

**Theorem 5 ([35])** For any graph $G$ with $\Delta \leq 3$ and $i$ isolated vertices,

$$\gamma(G) \leq \frac{1}{4}(3n - m + i).$$

Another well-known result involving order and size of a graph $G$ is due to Berge [9], and it was also mentioned in Vizing’s work [84].

**Theorem 6 ([9, 84])** For any graph $G$,

$$n - m \leq \gamma(G) \leq n + 1 - \sqrt{1 + 2m}.$$  

The following theorem relates the domination number with the size of a graph $G$.

**Theorem 7 ([76])** For a connected graph $G$ with $\delta \geq 2$,

$$\gamma(G) \leq (m + 2)/3.$$  

*The equality is achieved if and only if $G$ is a cycle of length $n \equiv 1 \mod 3$.  

Another well-known theorem on the domination number of a graph is due to Niemen [67]. Let $\varepsilon_F(G)$ denote the maximum number of pendant edges in a spanning forest of $G$. 


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**Theorem 8 ([67])** For any graph $G$,

$$\gamma(G) + \varepsilon_F(G) = n.$$ 

Niemen used the above result to derive upper and lower bounds for $\gamma(G)$ in terms of a simply constructed spanning forest of $G$.

The following result proved in [53, 71] is an analogue of Nordhaus and Gaddum’s bounds [68] for the chromatic number of a graph, and it gives sharp bounds on the sum and product of the domination numbers of a graph and its complement.

**Theorem 9 ([53, 71])** For any graph $G$,

$$\gamma(G) + \gamma(\overline{G}) \leq n + 1,$$

$$\gamma(G)\gamma(\overline{G}) \leq n$$

and these bounds are sharp.

The next theorem is devoted to random graphs. A random graph is a graph in which graph edges are determined randomly. The most commonly studied random graph model is the Erdős - Rényi model, where a random graph, denoted $G(n, p)$, is obtained by starting with a set of $n$ vertices, and every possible edge occurs independently with probability $p$, $0 \leq p \leq 1$. A graph property $\mathcal{P}$ is said to hold for almost every graph if the probability that a random graph $G \in \mathcal{G}(n, p)$ has property $\mathcal{P}$ has the limiting value of 1 as $n \to \infty$. 

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Theorem 10 ([85]) Let \( k = \lfloor (\log_2 n - 2 \log_2 \log_2 n + \log_2 \log_2 e) \rfloor \). Then for almost every graph \( G \) of order \( n \),

\[
k + 1 \leq \gamma(G) \leq k + 2.
\]

Bounds in Terms of Order and Maximum/Minimum Degrees

The following elementary bound for the domination number is given by Berge in [11].

Theorem 11 ([11]) For any graph \( G \) of order \( n \),

\[
\left\lceil \frac{n}{1 + \Delta} \right\rceil \leq \gamma(G) \leq n - \Delta.
\]

Flach and Volkmann [36] in their works on domination proved the following interesting result:

Theorem 12 ([36]) For any graph \( G \) of order \( n \) with \( \delta > 0 \),

\[
\gamma(G) \leq \frac{1}{2} \left( n + 1 - (\delta - 1) \frac{\Delta}{\delta} \right).
\]

This theorem easily implies the result of Payan [70]:

Corollary 1 ([70]) For any graph \( G \) of order \( n \) with no isolated vertex and minimum degree \( \delta \),

\[
\gamma(G) \leq \frac{1}{2}(n + 2 - \delta).
\]

The following fundamental result was proved by many authors, and it provides an excellent upper bound for \( \gamma(G) \) when \( \delta \) is big enough.
Theorem 13 ([3, 4, 61, 70]) \textit{For any graph }$G$,

\[ \gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n. \]  

(1.1)

Here is the sketch of the proof of Theorem 13.

We put

\[ p = \frac{\ln(\delta + 1)}{(\delta + 1)} \]

and select a set of vertices $A$, a subset of vertices of a graph $G$, so that each vertex is selected independently with the probability $p$. Let

\[ B = V(G) - N[A]. \]

It is very easy to see that $S = A \cup B$ is a dominating set. The expected value of $|A|$ is $np$.

The probability that a vertex $v$ is in $B$ is

\[ (1 - p)^{1 + \deg(v)} \leq (1 - p)^{1 + \delta} \leq e^{-p(1 + \delta)}. \]

Hence, the expectation of $|S|$ is

\[ \mathbf{E}(|S|) = \mathbf{E}(|A|) + \mathbf{E}(|B|) \leq np + ne^{-p(1+\delta)} \]

\[ = (p + e^{-p(1+\delta)})n = \frac{\ln(\delta + 1) + 1}{\delta + 1} n. \]

Therefore, there is a particular set $S$ with at most this cardinality.
Using probabilistic methods, Alon [2] proved that this bound is asymptotically best possible. More precisely, he proved that when \( n \) is large there exists a graph \( G \) such that
\[
\gamma(G) \geq \frac{\ln(\delta + 1) + 1}{\delta + 1} n(1 + o(1)).
\]

In the following theorems, the bound of Theorem 13 is slightly improved:

**Theorem 14 ([4, 70])** For any graph \( G \) with \( n \) vertices and minimum degree \( \delta \),
\[
\gamma(G) \leq \frac{n}{\delta + 1} \sum_{j=1}^{\delta+1} \frac{1}{j}.
\]

**Theorem 15 (Caro and Roditty, [48] p. 48)** For any graph \( G \) with \( \delta \geq 1 \),
\[
\gamma(G) \leq \left( 1 - \frac{\delta}{(\delta + 1)^{1+1/\delta}} \right) n.
\]

In the next chapters, we will show in detail how the above upper bounds can be generalized for multiple domination.

Theorems 13, 14 and 15 provide very good upper bounds when \( \delta \) is big enough. For small values of \( \delta \), there are better results:

**Theorem 16 ([69])** If \( G \) is a graph with \( n \) vertices and \( \delta \geq 1 \), then
\[
\gamma(G) \leq n/2.
\]

**Theorem 17 ([64])** If \( G \) is a connected graph with \( n \geq 8 \) and \( \delta \geq 2 \), then
\[
\gamma(G) \leq 2n/5.
\]
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Theorem 18 ([74]) If $G$ is a connected graph of order $n$ with $\delta \geq 3$, then

$$\gamma(G) \leq 3n/8,$$

and the bound is sharp.

1.3 Method Used in Research

The estimation method of first moments has been used in this research. When analyzing data structures with random inputs we are generally interested in quantities such as time complexities, storage requirements or the value of a particular parameter characterizing the algorithm. These quantities are random non-deterministic since either the input is assumed to vary or the algorithm itself makes random decisions. In many instances these quantities, though random, have a deterministic behaviour for large inputs. The first moment method is widely used to derive such relationships. This method has been described in detail by Tao and Vu [83]. The description of the method in general follows.

If $A$ is an event in some sample space, let us denote by $P(A)$ the probability of $A$. If $X$ is a real-valued random variable with discrete support, the expectation of $X$ is defined as follows:

$$E(X) = \sum_{x} xP(X = x).$$

The first moment method can be derived using Markov’s inequality ([62], 1889).
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Theorem 19 ([83], [62]) Let $X$ be a non-negative random variable. Then for any positive real $\lambda > 0$,

$$
P(X \geq \lambda) \leq \frac{E(X)}{\lambda}.
$$

It is easy to see that if a random variable $X$ is integer-valued, and $\lambda$ is equal to 1, then the following inequality will hold true:

$$
P(X > 0) \leq E(X).
$$

This is called the first moment method. The first moment method can also be derived without referring to Markov’s inequality. Observing that

$$
P(X > 0) = \sum_{k=1}^{\infty} P(X = k) \leq \sum_{k=0}^{\infty} kP(X = k) = E(X),
$$

the inequality (1.2) is obtained.

To apply the first moment method it is necessary to find a ‘good’ upper bound for the expectations of a random variable. A fundamental tool in doing so is linearity of expectation, which states that

$$
E(c_1X_1 + ... + c_nX_n) = c_1E(X_1) + ... + c_nE(X_n),
$$

whenever $X_1, ..., X_n$ are random variables and $c_1, ..., c_n$ are real numbers. The main point of this principle comes from the fact that there is no restriction on the independence or dependence among the $X_i$-s.

The following is an adjustment of the first moment method that is used in this thesis. The method is adjusted for finding upper bounds for various domination
1.4 Thesis Organization

parameters.

1. Let $A$ be a set, which is randomly generated by an independent choice of vertices of $G$, where each vertex is selected with the probability $p$.

2. Using the set $A$, a set $D$ is constructed in such a way that it has properties necessary for carrying out the estimation (for instance, $D$ is a dominating set).

3. The random variable $|D|$ (size of the set $D$) is examined and an upper bound of its expectation is derived: $E(|D|) \leq f(.)$. The parameters of the function $f(.)$ may vary depending on the problem under consideration.

4. Because expectation is an average value, it can be concluded that there exists a particular set $D$, which satisfies the following condition: $|D| \leq f(.)$, thus providing the required upper bound.

1.4 Thesis Organization

Domination is the main object of study in this thesis. Using a probabilistic approach, some new upper bounds for various domination parameters have been derived and are presented in the chapters of the thesis. Each chapter of this thesis is self contained, i.e. it has been submitted for publication in a scientific journal.

The organization of the thesis is explained below.

Chapter 1 is the introduction to this thesis. In this chapter, the background to this work, preliminaries, notions, the concept of domination and its applications for real life problems are discussed. Moreover, the review of existing results is
provided. Also, the method, which was used in this thesis to derive new upper bounds for domination parameters is described in detail.

In Chapter 2, the notions of $k$-domination and $k$-tuple domination in graphs are discussed in detail, and new upper bounds as well as well-known ones are presented. Also, the proofs for the new results are presented. The new results imply Rautenbach-Volkmann’s conjecture and an analogue of this conjecture for the $k$-domination number.

In Chapter 3, a new upper bound for the $\alpha$-domination number is provided. This result generalizes the well-known Caro-Roditty bound for the domination number of a graph. The same probabilistic construction is used to generalize another well-known upper bound for the classical domination in graphs. Also, similar upper bounds for the $\alpha$-rate domination number, which combines the concepts of $\alpha$-domination and $k$-tuple domination are presented.

In Chapter 4, new upper bounds for the global domination and Roman domination numbers are presented and it is also proved that these results are asymptotically best possible. Moreover, in this chapter, the upper bounds for the restrained domination and total restrained domination numbers for large classes of graphs are given, and it is shown that, for almost all graphs, the restrained domination number is equal to the domination number, and the total restrained domination number is equal to the total domination number. A number of open problems are posed.

In Chapter 5, the concepts of signed domination and discrepancy are discussed and new upper and lower bounds for the signed domination number are presented. These new bounds improve a number of known results, which are discussed in the chapter too.
Chapter 6 summarizes and concludes the work presented in this thesis as well as suggests possible directions for future research.
Chapter 2

On $k$-Domination and $k$-Tuple Domination in Graphs

2.1 Introduction

In this chapter, $k$-domination, $k$-tuple domination and $k$-total domination in graphs are discussed in detail. A review of well-known results is given and the new results with the proofs used to derive them are presented. These upper bounds generalize two classical bounds for the single domination number and also improve a number of known bounds for the $k$-domination and $k$-tuple domination parameters. Effective randomized algorithms for finding corresponding $k$-dominating and $k$-tuple dominating sets, whose expected sizes satisfy the upper bounds presented are provided. These algorithms follow from the probabilistic constructions used to prove the corresponding theorems. The algorithms are linear in the number of edges of the input graph. Also, they can be implemented in parallel or as local distributed algorithms. The corresponding multiple domination problems
are known to be NP-complete.

Throughout this chapter, a concept of $m$-degree of a graph $G$ is used. For $m \leq \delta$, Gagarin and Zverovich [38] defined the $m$-degree of a graph $G$ as follows:

$$\hat{d}_m = \hat{d}_m(G) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d_i}{m} \right).$$

Note that $\hat{d}_1$ is the average degree $d$ of a graph, $\hat{d}_0 = 1$ and $\hat{d}_{-1} = 0$.

### 2.1.1 $k$-Domination

The concept of $k$-domination was introduced by Fink and Jacobson [34]. A set $X$ is called a $k$-dominating set if every vertex not in $X$ has at least $k$ neighbours in $X$. The minimum cardinality of a $k$-dominating set of $G$ is the $k$-domination number $\gamma_k(G)$. It is easy to see that $\gamma_1(G) = \gamma(G)$. Obviously, every $(k+1)$-dominating set is a $k$-dominating set, i.e. $\gamma_k(G) \leq \gamma_{k+1}(G)$. Also, the vertex set $V$ is the only $(\Delta+1)$-dominating set. However, it is not a minimum $\Delta$-dominating set. Thus, it can be observed (see [34]) that for every graph $G$ the following is true:

$$\gamma(G) = \gamma_1(G) \leq \gamma_2(G) \leq \ldots \leq \gamma_{\Delta}(G) < \gamma_{\Delta+1}(G) = |V|.$$ 

It was shown by Jacobson and Peters [48] that the problem of finding a minimum $k$-dominating set is NP-hard. Thus, in order to understand this concept better, it is of importance to analyze the upper bounds of this parameter.

The first upper bound for the $k$-domination number is due to Cockayne, Gamble and Shepherd [23].
2.1 Introduction

**Theorem 20 ([23])** For any graph $G$ with $\delta \geq k$,

$$\gamma_k(G) \leq \frac{k}{k+1}n.$$

Caro and Roditty [18] and Stracke and Volkmann [81] improved the above bound. In particular, they proved the following:

**Theorem 21**

(a) [81] $\gamma_k(G) \leq \frac{2k-\delta}{2k-\delta+1}n$ if $k \leq \delta \leq 2k-1$;

(b) [18, 81] $\gamma_k(G) \leq \frac{n}{2}$ if $\delta \geq 2k-1$.

The proof of Theorem 25 implies the following non-asymptotic bound, which improves the above results in the case when $\delta$ is ‘much bigger’ than $k$.

**Theorem 22 ([17])** If $\delta > e^{k^2}$, then

$$\gamma_k(G) < \frac{\ln \delta + 2}{\delta}n.$$

Rautenbach and Volkmann [72] extended this result for smaller values of $\delta$:

**Theorem 23 ([72])** If $\delta \geq 2k\ln(\delta + 1) - 1$, then

$$\gamma_k(G) \leq \frac{k\ln(\delta + 1) + \sum_{i=0}^{k-1} \frac{1}{\delta(\delta+1)^{k-1-i}}}{\delta + 1}n.$$
\( \beta(G) \) and is called the covering number of \( G \). Note that interesting bounds for the 2-domination number can be found in \([13]\).

**Theorem 24** ([13]) *If \( G \) is a graph with \( \delta(G) \geq 2 \), then every covering is also a 2-dominating set and thus \( \gamma_2(G) \leq \beta(G) \).*

### 2.1.2 \( k \)-Tuple and \( k \)-Total Domination

Another domination parameter, the \( k \)-tuple domination was introduced in \([45]\). A set \( X \) is called a \( k \)-tuple dominating set of \( G \) if for every vertex \( v \in V(G) \), \( |N[v] \cap X| \geq k \). The minimum cardinality of a \( k \)-tuple dominating set of \( G \) is the \( k \)-tuple domination number \( \gamma_{xk}(G) \). The \( k \)-tuple domination number is only defined for graphs with \( \delta \geq k - 1 \). It is easy to see that \( \gamma(G) = \gamma_{x1}(G) \) and \( \gamma_{xk}(G) \leq \gamma_{xk'}(G) \) for \( k \leq k' \). The 2-tuple domination number \( \gamma_{x2}(G) \) is called the double domination number and the 3-tuple domination number \( \gamma_{x3}(G) \) is called the triple domination number. The \( k \)-total domination number is a slightly different domination parameter, and it was studied by Kulli \([58]\). A set \( X \) is called a \( k \)-total dominating set of \( G \) if every vertex \( v \in V(G) \) is dominated by at least \( k \) vertices in \( X \setminus \{v\} \). The minimum cardinality of a \( k \)-total dominating set of \( G \) is the \( k \)-total domination number \( \gamma_{tk}^k(G) \). Note that a \( k \)-total dominating set is a \( k \)-tuple dominating set but not vice versa.

Caro and Yuster \([17]\) proved an interesting asymptotic result that if \( \delta \) is ‘much bigger’ than \( k \), then the upper bound for the \( k \)-total domination number is ‘close’ to the bound of Theorem \([13]\). More precisely, they proved the following:
Theorem 25 ([17]) If $\delta > e^k$, then

$$\gamma_k(G) \leq \frac{\ln \delta}{\delta} n(1 + o_\delta(1)).$$

The same upper bound is therefore true for the $k$-tuple domination and $k$-domination numbers.

For large values of $\delta$, the proof of Theorem 25 implies the following strong upper bound for the $k$-tuple and $k$-total domination numbers:

Theorem 26 ([17]) If $\delta > e^k$, then

$$\gamma_{\times k}(G) \leq \gamma_k(G) < \frac{\ln \delta + \sqrt{\ln \delta} + 2}{\delta} n.$$

This is the only known upper bound for the $k$-total domination number. For small values of $\delta$, a number of authors gave upper bounds for the $k$-tuple domination number. Harant and Henning [42] found an upper bound for the double domination number:

Theorem 27 ([42]) For any graph $G$ with $\delta \geq 1$,

$$\gamma_{\times 2}(G) \leq \frac{\ln \delta + \ln(d + 1) + 1}{\delta} n.$$

An interesting upper bound for the triple domination number was given by Rautenbach and Volkmann [72]:

Theorem 28 ([72]) For any graph $G$ with $\delta \geq 2$,

$$\gamma_{\times 3}(G) \leq \frac{\ln(\delta - 1) + \ln(\hat{d}_2 + d) + 1}{\delta - 1} n.$$
They also proved an upper bound similar to that of Theorem 23:

**Theorem 29 ([72])** If $\delta \geq 2k \ln(\delta + 1) - 1$, then

$$\gamma_k(G) \leq \frac{k \ln(\delta + 1) + \sum_{i=0}^{k-1} \frac{k-i}{\delta(\delta+1)^{k-i-1}}}{\delta + 1} n.$$ 

The following generalization of the above upper bounds for double and triple domination was posed as a conjecture by Rautenbach and Volkmann [72]:

**Conjecture 1 ([72])** For any graph $G$ with $\delta \geq k - 1$,

$$\gamma_k(G) \leq \frac{\ln(\delta - k + 2) + \ln(\hat{d}_{k-1} + \hat{d}_{k-2}) + 1}{\delta - k + 2} n.$$ 

This conjecture was recently proved by Zverovich [87, 88] and also by Chang [19]. Both proofs exploit the idea of randomly generating a $k$-tuple dominating set from [38]. Notice that Chang’s proof [19] is not entirely correct. The author claims that for $0 \leq m \leq k - 1$,

$$\left( d_i + 1 \right)^m \left( d_i + 1 - m \right) \geq \left( d_i + 1 \right)^m \left( d_i + 1 - m \right), \quad (2.1)$$

which follows from the fact that $d_i + 1 \geq \delta + 1 \geq k$. However, inequality (2.1) is not correct. For example, if $m = k - 1$, then $\left( d_i + 1 - m \right) = 1$ while $d_i + 1 - m = d_i + 2 - k > 1$ if $d_i + 1 > k$.

The new upper bound on $k$-tuple domination presented in this chapter improves the upper bound of Conjecture 1.
2.2 New Upper Bounds for the $k$-Tuple Domination Number

The following theorem improves the upper bound of Conjecture 1.

**Theorem 30** For any graph $G$ with $\delta \geq k$,

$$\gamma_{\times k}(G) \leq \left(1 - \frac{\delta'}{d_{k-1} + d_{k-2})^{1/\delta'}(1 + \delta')^{1+1/\delta'}}\right)^n,$$

where $\delta' = \delta - k + 1$.

**Proof:** Let

$$p = 1 - 1/\left((d_{k-1} + d_{k-2})(1 + \delta')\right)^{1/\delta'}$$

and let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with the probability $p$. For $m = 0, 1, \ldots, k - 1$, let us denote

$$B_m = \{v_i \in V(G) - A : |N(v_i) \cap A| = m\}.$$

Also, for $m = 0, 1, \ldots, k - 2$, we denote

$$A_m = \{v_i \in A : |N(v_i) \cap A| = m\}.$$

For each set $A_m$, we form a set $A'_m$ in the following way. For every vertex in the set $A_m$, we take $k - m - 1$ neighbours not in $A$ and add them to $A'_m$. Such neighbours always exist because $\delta \geq k$. It is obvious that

$$|A'_m| \leq (k - m - 1)|A_m|.$$
2.2 New Upper Bounds for the $k$-Tuple Domination Number

For each set $B_m$, we form a set $B'_m$ by taking $k - m - 1$ neighbours not in $A$ for every vertex in $B_m$. We have

$$|B'_m| \leq (k - m - 1)|B_m|.$$  

We construct the set $D$ as follows:

$$D = A \cup \left( \bigcup_{m=0}^{k-2} A'_m \right) \cup \left( \bigcup_{m=0}^{k-1} B_m \cup B'_m \right).$$

It is easy to see that $D$ is a $k$-tuple dominating set. The expectation of $|D|$ is

$$E(|D|) \leq E\left(|A| + \sum_{m=0}^{k-2} |A'_m| + \sum_{m=0}^{k-1} |B_m| + \sum_{m=0}^{k-1} |B'_m|\right)$$

$$\leq E(|A|) + \sum_{m=0}^{k-2} (k - m - 1)E(|A_m|) + \sum_{m=0}^{k-1} (k - m)E(|B_m|).$$

We have

$$E(|A_m|) = \sum_{i=1}^{n} P(v_i \in A_m)$$

$$= \sum_{i=1}^{n} p \binom{d_i}{m} p^m (1 - p)^{d_i - m}$$

$$\leq p^{m+1}(1 - p)^{d_{\hat{m}}} \gamma_{\hat{m}n}$$

and

$$E(|B_m|) = \sum_{i=1}^{n} P(v_i \in B_m)$$

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2.2 New Upper Bounds for the $k$-Tuple Domination Number

\[
\begin{align*}
&= \sum_{i=1}^{n} (1-p) \binom{d_i}{m} p^m (1-p)^{d_i - m} \\
&\leq p^m (1-p)^{\delta - m + 1} \hat{d}_m n.
\end{align*}
\]

Taking into account that \( \hat{d}_{-1} = 0 \), we obtain

\[
\mathbb{E}( |D| ) \leq pn + \sum_{m=0}^{k-2} (k - m - 1) p^m (1-p)^{\delta - m} \hat{d}_m n + \sum_{m=0}^{k-1} (k - m) p^m (1-p)^{\delta - m + 1} \hat{d}_m n = pn + \sum_{m=1}^{k-1} (k - m) p^m (1-p)^{\delta - m + 1} \hat{d}_m n + \sum_{m=0}^{k-1} (k - m) p^m (1-p)^{\delta - m + 1} \hat{d}_m n = pn + (1-p)^{\delta - k + 2} \sum_{m=0}^{k-1} (k - m) p^m (1-p)^{k - m - 1} (\hat{d}_{m-1} + \hat{d}_m).
\]

Furthermore, for $0 \leq m \leq k - 1$,\[
(k - m)(\hat{d}_{m-1} + \hat{d}_m) = \sum_{i=1}^{n} (k - m) \binom{d_i + 1}{m} / n \leq \sum_{i=1}^{n} \frac{d_i - k + 2}{j} \frac{(k - m + j - 1)}{j} \binom{d_i + 1}{m} / n = \sum_{i=1}^{n} \frac{(d_i - m + 1)}{d_i - k + 2} \frac{(d_i + 1)}{m} / n = \sum_{i=1}^{n} \frac{(k - 1)}{m} \frac{(d_i + 1)}{k - 1} / n = \binom{k - 1}{m} (\hat{d}_{k-1} + \hat{d}_{k-2}).
\]

We obtain

\[
\mathbb{E}( |D| ) \leq pn + (1-p)^{\delta + 1} n (\hat{d}_{k-1} + \hat{d}_{k-2}) \sum_{m=0}^{k-1} \frac{(k - 1)}{m} p^m (1-p)^{k - m - 1} = pn + (1-p)^{\delta + 1} n (\hat{d}_{k-1} + \hat{d}_{k-2})
\]

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2.3 New Upper Bounds for the $k$-Domination Number

Rautenbach and Volkmann [72] found an interesting upper bound for the $k$-domination number under the assumption that $\delta > 2k \ln(\delta+1) - 1$. The following results provide similar upper bounds for $\delta \geq k$:

$$\leq \left(1 - \frac{\delta'}{\hat{d}_{k-1} + \hat{d}_{k-2}}^{1/\delta'}(1 + \delta')^{1/\delta'}\right)n,$$

as required. The proof of the theorem is complete.

Theorem 30 implies Conjecture 1.

**Corollary 2** For any graph $G$ with $\delta \geq k - 1$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\hat{d}_{k-1} + \hat{d}_{k-2}) + 1}{\delta - k + 2}n.$$

**Proof:** Using the inequality $1 - p \leq e^{-p}$, the proof of Theorem 30 implies a weaker upper bound for $E(|D|)$:

$$E(|D|) \leq pm + e^{-p(\delta+1)}n(\hat{d}_{k-1} + \hat{d}_{k-2}).$$

The result easily follows if we put $p = \min\{1, \frac{\ln(\delta' + 1) + \ln(\hat{d}_{k-1} + \hat{d}_{k-2})}{\delta' + 1}\}$. 

In some cases, Theorem 30 provides a much better upper bound than Corollary 2 does. For example, let $G$ be a 20-regular graph. Then, according to Corollary 2, $\gamma_{\times 5}(G) < 0.738n$, while Theorem 30 yields $\gamma_{\times 5}(G) < 0.543n$.
2.3 New Upper Bounds for the $k$-Domination Number

**Theorem 31**  For any graph $G$ with $\delta \geq k$,

$$
\gamma_k(G) \leq \left(1 - \frac{\delta'}{\hat{d}_{k-1}^{1/\delta'} (1 + \delta')^{1+1/\delta'}}\right)n,
$$

where $\delta' = \delta - k + 1$.

**Proof:** Let

$$p = 1 - 1/\left(\hat{d}_{k-1}(1 + \delta')^{1/\delta'}\right)
$$

and let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with the probability $p$. For $m = 0, 1, ..., k-1$, let us denote

$$B_m = \{v_i \in V(G) - A : |N(v_i) \cap A| = m\}.$$

We construct the set $D$ as follows:

$$D = A \cup \left(\bigcup_{m=0}^{k-1} B_m\right).$$

It is easy to see that $D$ is a $k$-dominating set. The expectation of $|D|$ is

$$E(|D|) \leq E\left(|A| + \sum_{m=0}^{k-1} |B_m|\right) = E(|A|) + \sum_{m=0}^{k-1} E(|B_m|).$$

We have

$$E(|B_m|) = \sum_{i=1}^{n} P(v_i \in B_m)$$

$$= \sum_{i=1}^{n} (1-p)^m (d_i^{-m}) p^m (1-p)^{d_i-m}$$
2.3 New Upper Bounds for the $k$-Domination Number

$$\leq p^m(1-p)^{\delta-m+1}\hat{d}_m n.$$ 

Therefore,

$$E(|D|) \leq pn + \sum_{m=0}^{k-1} p^m(1-p)^{\delta-m+1}\hat{d}_m n$$

$$= pn + (1-p)^{\delta-k+2} n \sum_{m=0}^{k-1} p^m (1-p)^{k-m-1}\hat{d}_m.$$ 

Furthermore, for $0 \leq m \leq k-1$,

$$\hat{d}_m = \sum_{i=1}^{n} \binom{d_i}{m} / n$$

$$\leq \sum_{i=1}^{n} \binom{d_i - m}{d_i - k + 1} \binom{d_i}{m} / n$$

$$= \sum_{i=1}^{n} \binom{k-1}{m} \binom{d_i}{k-1} / n$$

$$= \binom{k-1}{m} \hat{d}_{k-1}.$$ 

We obtain

$$E(|D|) \leq pn + (1-p)^{\delta+1} n \hat{d}_{k-1} \sum_{m=0}^{k-1} \binom{k-1}{m} p^m (1-p)^{k-m-1}$$

$$= pn + (1-p)^{\delta+1} n \hat{d}_{k-1}$$

$$\leq \left(1 - \frac{\delta'}{\hat{d}_{k-1}^{1/\delta'}} (1 + \delta')^{1+1/\delta'} \right) n,$$

as required. The proof of Theorem 31 is complete.

An analogue of Rautenbach–Volkmann’s conjecture for the $k$-domination number follows from Theorem 31.
2.4 Effective Randomized Domination

**Corollary 3** For any graph $G$ with $\delta \geq k$,

$$\gamma_k(G) \leq \frac{\ln(\delta - k + 2) + \ln \hat{d}_{k-1} + 1}{\delta - k + 2} n.$$

**Proof:** The proof is similar to one of Corollary 2.

If $\Delta$ is ‘close’ to $\delta$, then Corollary 3 can give a better bound than Theorem 23. For example, let $G$ be a 100-regular graph. Then, by Corollary 3, $\gamma_{10}(G) < 0.368n$, while Theorem 23 yields $\gamma_{10}(G) < 0.457n$. However, if $\Delta$ is ‘much bigger’ than $\delta$, then Theorem 23 becomes better.

**Corollary 4 (Caro and Roditty, [48] p.48)** For any graph $G$ with $\delta \geq 1$,

$$\gamma(G) \leq \left(1 - \frac{\delta}{(1 + \delta)^{1+1/\delta}}\right)n.$$

**Proof:** The proof follows from either Theorem 30 or Theorem 31 if we put $k = 1$.

---

2.4 Effective Randomized Domination

In this section, we present randomized algorithms for constructing $k$-dominating and $k$-tuple dominating sets, whose expected sizes satisfy the upper bounds of Theorems 30 and 31. The algorithms follow from probabilistic constructions used to prove the corresponding theorems. These results generalize the classical upper bound for the single domination (Theorem 13) and improve a number of known upper bounds for the $k$-tuple domination number as described above. Notice that
2.4 Effective Randomized Domination

a simple deterministic algorithm to construct a dominating set satisfying bound \(\text{(1.1)}\) can be found in \[3\].

2.4.1 \(k\)-Tuple Domination

The probabilistic construction used in the proof of Theorem \[30\] implies randomized Algorithm \[1\] to find a \(k\)-tuple dominating set, whose size satisfies the bound of Theorem \[30\] with a positive probability. In other words, the expectation of the size of the set \(D\) returned by Algorithm \[1\] satisfies the upper bound of Theorem \[30\].

**Algorithm 1**: Randomized \(k\)-tuple dominating set

**Input**: A graph \(G\) and an integer \(k\), \(k \leq \delta\).

**Output**: A \(k\)-tuple dominating set \(D\) of \(G\).

begin
    Compute \(p = 1 - 1/\left(\hat{d}_{k-1} + \hat{d}_{k-2}(1 + \hat{\delta}')\right)^{1/\hat{\delta}'}\);
    Initialize \(A = \emptyset\); /* Form a set \(A \subseteq V(G)\) */
    foreach vertex \(v \in V(G)\) do
        with the probability \(p\), decide if \(v \in A\) or \(v \notin A\);
    end
    Initialize \(B = \emptyset\); /* Form a set \(B \subseteq V(G) - A\) */
    foreach vertex \(v \in V(G)\) do
        Compute \(r = |N[v] \cap A|\);
        if \(r < k\) then
            if \(v \in A\) then
                add any \(k - r\) vertices from \(N(v) - A\) into \(B\);
            else /* \(v \notin A\) */
                add \(v\) and any \(k - r - 1\) vertices from \(N(v) - A\) into \(B\);
            end
        end
    end
    Put \(D = A \cup B\); /* \(D\) is a \(k\)-tuple dominating set */
    return \(D\);
end
2.4 Effective Randomized Domination

2.4.2 $k$-Domination

Algorithm 2 given below presents a randomized algorithm to find a $k$-dominating set whose size satisfies the upper bound of Theorem 31 with a positive probability. The algorithm is based on the probabilistic construction used in the proof of Theorem 31 and the expectation of the size of the set $D$ returned by Algorithm 2 satisfies the upper bound of Theorem 31.

**Algorithm 2**: Randomized $k$-dominating set

**Input**: A graph $G$ and an integer $k$, $k \leq \delta$.

**Output**: A $k$-dominating set $D$ of $G$. 

begin

Compute $p = 1 - 1/(\hat{\delta}_k - 1 + \delta')^{1/\delta'}$;

Initialize $A = \emptyset$; /* Form a set $A \subseteq V(G)$ */

foreach vertex $v \in V(G)$ do

| with the probability $p$, decide if $v \in A$ or $v /\in A$;

end

Initialize $B = \emptyset$; /* Form a set $B \subseteq V(G) - A$ */

foreach vertex $v \in V(G) - A$ do

| if $|N(v) \cap A| < k$ then

| /* $v$ is dominated by fewer than $k$ vertices of $A$ */

| add $v$ into $B$;

end

end

Put $D = A \cup B$; /* $D$ is a $k$-dominating set */

return $D$;

end

2.4.3 Complexity and Implementation

Assuming that an input graph $G$ has no isolated vertex, the minimum vertex degree $\delta$ of $G$ or its degree sequence can be computed in $O(m)$ time, where $m$ is
2.4 Effective Randomized Domination

the number of edges in $G$. Then Algorithm 1 can take up to $O(m)$ time. More precisely, in reference to Algorithm 1 a worst case scenario when $k$ is close to $\delta/2$ may require $O(\delta)$ steps to compute $\tilde{d}_{k-1}$, and $\delta'$ can be computed in $O(1)$. Therefore, in total it takes $O(\delta)$ time to compute the probability $p$. Clearly, it takes $O(n)$ time to find the set $A$. The numbers $r = |N[v] \cap A|$ for each $v \in V(G)$ can be computed separately or when finding the set $A$. In any case, we need to keep track of them only up to $r = k$. Since we may need to browse through all the neighbours of vertices in $A$, in total it can take $O(m)$ steps to calculate all the necessary $r$’s for each vertex $v \in V(G)$. Then the set $B$ can be also found in $O(m)$ steps. Thus, in total Algorithm 1 runs in $O(m)$ time.

For Algorithm 2 a complexity analysis similar to that of Algorithm 1 shows that it can take up to $O(m)$ steps to find a $k$-dominating set.

Algorithms 1 and 2 are presented here in a form consistent with the proofs of the corresponding theorems. However, when implementing these algorithms, the output sets $D$ can be constructed more efficiently and effectively by a recursive extension of the corresponding initial set $A$. In other words, instead of adding the necessary vertices into sets $B$, we can add them directly into $A$. This can result in a smaller $k$-tuple ($k$-dominating) set $D$. 
Chapter 3

Upper Bounds for $\alpha$-Domination Parameters

3.1 Introduction

In this chapter\(^1\), $\alpha$-domination and $\alpha$-rate domination are discussed in detail, and a new upper bound for the $\alpha$-domination number is proved by using a probabilistic construction. This result generalizes the well-known classical upper bounds for the domination number (see Theorem 13 and Theorem 15). Similar upper bounds are proved for the $\alpha$-rate domination number, which combines the concepts of $\alpha$-domination and $k$-tuple domination. Also, effective randomized algorithms for finding $\alpha$-dominating and $\alpha$-rate dominating sets, whose expected sizes satisfy the upper bounds, are presented.

\(^1\)Parts of the work given in this chapter are based on a journal paper [39] included in Appendix A.
3.1 Introduction

3.1.1 $\alpha$-Domination

The $\alpha$-domination was introduced by Dunbar et al. in [30]. The concept of $\alpha$-domination is different from the concept of $k$-domination in that a vertex must be dominated by a percentage of the vertices in its neighbourhood instead of a fixed number of its neighbours. Let $\alpha$ be a real number satisfying $0 < \alpha \leq 1$. A set $X \subseteq V(G)$ is called an $\alpha$-dominating set of $G$ if

$$|N(v) \cap X| \geq \alpha d_v$$

for every vertex $v \in V(G) - X$, i.e. $v$ is adjacent to at least $[\alpha d_v]$ vertices of $X$. The minimum cardinality of an $\alpha$-dominating set of $G$ is called the $\alpha$-domination number $\gamma_\alpha(G)$. It is easy to see that $\gamma(G) \leq \gamma_\alpha(G)$, and

$$\gamma_{\alpha_1}(G) \leq \gamma_{\alpha_2}(G) \text{ for } \alpha_1 < \alpha_2.$$

Also, $\gamma(G) = \gamma_\alpha(G)$ if $\alpha$ is sufficiently close to 0.

For $0 < \alpha \leq 1$, the $\alpha$-degree of a graph $G$ is defined as follows:

$$\hat{d}_\alpha = \hat{d}_\alpha(G) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d_i}{\lceil \alpha d_i \rceil} - 1 \right).$$

The following results are proved in [30]:

$$\frac{\alpha \delta n}{\Delta + \alpha \delta} \leq \gamma_\alpha(G) \leq \frac{\Delta n}{\Delta + (1 - \alpha) \delta}$$

(3.1)

and

$$\frac{2\alpha m}{(1 + \alpha)\Delta} \leq \gamma_\alpha(G) \leq \frac{(2 - \alpha)\Delta n - (2 - 2\alpha)m}{(2 - \alpha)\Delta}.$$

(3.2)
3.1 Introduction

Interesting results on $\alpha$-domination perfect graphs can be found in [26]. The problem of deciding whether $\gamma_\alpha(G) \leq k$ for a positive integer $k$ is known to be NP-complete [30]. Therefore, it is important to have good upper bounds for the $\alpha$-domination number and efficient approximation algorithms for finding ‘small’ $\alpha$-dominating sets. In this chapter, new upper bounds and the corresponding effective randomized algorithms are presented.

3.1.2 $\alpha$-Rate Domination

The concept of $\alpha$-rate domination is similar to the well-known concept of $k$-tuple domination, and an $\alpha$-rate dominating set can be considered as a particular case of an $\alpha$-dominating set in the same graph. It will be shown that the $\alpha$-rate domination number satisfies upper bounds similar to those for $\alpha$-domination.

Let us define a set $X \subseteq V(G)$ to be an $\alpha$-rate dominating set of $G$ if for any vertex $v \in V(G)$,

$$ |N[v] \cap X| \geq \alpha d_v. $$

The minimum cardinality of an $\alpha$-rate dominating set of $G$ is called the $\alpha$-rate domination number $\gamma_{\times \alpha}(G)$. It is easy to see that $\gamma_\alpha(G) \leq \gamma_{\times \alpha}(G)$. For $0 < \alpha \leq 1$, the closed $\alpha$-degree of a graph $G$ is defined as follows:

$$ \tilde{d}_\alpha = \tilde{d}_\alpha(G) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d_i + 1}{\lceil \alpha d_i \rceil} - 1 \right). $$

In fact, the only difference between the $\alpha$-degree and the closed $\alpha$-degree is that to compute the latter we choose from $d_i + 1$ vertices instead of $d_i$, i.e. from the closed neighbourhood $N[v_i]$ of $v_i$ instead of $N(v_i)$. 

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3.2 New Upper Bounds for the $\alpha$-Domination Number

One of the strongest known upper bounds for the domination number is due to Caro and Roditty, see Theorem \[15\]. This upper bound is generalized for the $\alpha$-domination number in Theorem \[32\]. Indeed, if $d_i$ are fixed for all $i = 1, \ldots, n$, and $\alpha$ is sufficiently close to 0, then $\widehat{\delta} = \delta$ (provided $\delta \geq 1$) and $\widehat{d}_\alpha = 1$.

**Theorem 32** For any graph $G$,

$$
\gamma_\alpha(G) \leq \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1/\widehat{d}_\alpha}}\right)^n,
$$

where $\widehat{\delta} = \lfloor \delta(1 - \alpha) \rfloor + 1$.

**Proof:** Let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with the probability

$$
p = 1 - \left(\frac{1}{(1 + \widehat{\delta})d_\alpha}\right)^{1/\widehat{\delta}}.
$$

Let us denote

$$
B = \{v_i \in V(G) - A : |N(v_i) \cap A| \leq \lceil \alpha d_i \rceil - 1\}.
$$

It is obvious that the set $D = A \cup B$ is an $\alpha$-dominating set. The expectation of $|D|$ is

$$
\mathbf{E}(|D|) = \mathbf{E}(|A|) + \mathbf{E}(|B|)
$$
3.2 New Upper Bounds for the $\alpha$-Domination Number

\[= \sum_{i=1}^{n} P(v_i \in A) + \sum_{i=1}^{n} P(v_i \in B)\]

\[= pn + \sum_{i=1}^{n} \sum_{r=0}^{[\alpha d_i]-1} \binom{d_i}{r} p^r (1-p)^{d_i-r+1}. \]

It is easy to see that, for $0 \leq r \leq [\alpha d_i] - 1,$

\[\binom{d_i}{r} \leq \binom{d_i}{[\alpha d_i] - 1} \binom{[\alpha d_i] - 1}{r}.\]

Also,

\[d_i - [\alpha d_i] \geq \lfloor \delta (1-\alpha) \rfloor.\]

Therefore,

\[E(|D|) \leq pn + \sum_{i=1}^{n} \binom{d_i}{[\alpha d_i] - 1} (1-p)^{d_i-[\alpha d_i]+2} \times \]

\[\sum_{r=0}^{[\alpha d_i]-1} \binom{[\alpha d_i] - 1}{r} p^r (1-p)^{[\alpha d_i]-1-r} \]

\[= pn + \sum_{i=1}^{n} \binom{d_i}{[\alpha d_i] - 1} (1-p)^{d_i-[\alpha d_i]+2} \]

\[\leq pn + (1-p)^{\hat{\delta} (1-\alpha) + 2} \hat{\delta} n \]

\[= pn + (1-p)^{\hat{\delta} + 1} \hat{\delta} n \quad (3.5)\]

\[= \left(1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1+1/\hat{\delta}} \hat{\delta}^{1/\hat{\delta}} \hat{\delta}} \right) n.\]

Note that the value of $p$ in (3.4) is chosen to minimize the expression (3.5). Since the expectation is an average value, there exists a particular $\alpha$-dominating set of size at most \(1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1+1/\hat{\delta}} \hat{\delta}^{1/\hat{\delta}} \hat{\delta}} \) n, as required. The proof of the theorem is complete.
3.3 New Upper Bounds for \(\alpha\)-Rate Domination

Notice that in some cases Theorem 32 provides a much better bound than the upper bound in (3.1). For example, if \(G\) is a 1000-regular graph, then Theorem 32 gives \(\gamma_{0.1}(G) < 0.305n\), while (3.1) yields only \(\gamma_{0.1}(G) < 0.527n\).

**Corollary 5** For any graph \(G\),

\[
\gamma_{\alpha}(G) \leq \frac{\ln(\hat{\delta} + 1) + \ln \hat{d}_{\alpha} + 1}{\hat{\delta} + 1} n.
\] (3.6)

**Proof:** We put

\[
p = \min \left\{ 1, \frac{\ln(\hat{\delta} + 1) + \ln \hat{d}_{\alpha}}{\hat{\delta} + 1} \right\}.
\]

Using the inequality \(1 - p \leq e^{-p}\), we can estimate the expression (3.5) as follows:

\[
\mathbb{E}(|D|) \leq pn + e^{-p(\hat{\delta} + 1)}\hat{d}_{\alpha} n.
\]

If \(p = 1\), then the result easily follows. If \(p = \frac{\ln(\hat{\delta} + 1) + \ln \hat{d}_{\alpha}}{\hat{\delta} + 1}\), then

\[
\mathbb{E}(|D|) \leq \frac{\ln(\hat{\delta} + 1) + \ln \hat{d}_{\alpha} + 1}{\hat{\delta} + 1} n,
\]

as required.

Corollary 5 generalizes the well-known upper bound of Theorem 13, independently proved by several authors in \([3, 4, 61, 70]\).

### 3.3 New Upper Bounds for \(\alpha\)-Rate Domination

The following theorem provides an analogue of the Caro-Roditty bound (Theorem 15) for the \(\alpha\)-rate domination number:
3.3 New Upper Bounds for $\alpha$-Rate Domination

**Theorem 33** For any graph $G$ and $0 < \alpha \leq 1$,

$$\gamma_{\times \alpha}(G) \leq \left(1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1/\delta} d_{\alpha}^{1/\delta}} \right) n,$$

(3.7)

where $\hat{\delta} = [\delta(1 - \alpha)] + 1$.

**Proof:** Let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with probability $p$, $0 \leq p \leq 1$. For $m \geq 0$, denote by $B_m$ the set of vertices $v \in V(G)$ dominated by exactly $m$ vertices of $A$ and such that $|N[v] \cap A| < \alpha d_v$, i.e.

$$|N[v] \cap A| = m \leq \lceil \alpha d_v \rceil - 1.$$

Note that each vertex $v \in V(G)$ is in at most one of the sets $B_m$ and $0 \leq m \leq \lceil \alpha d_v \rceil - 1$. We form a set $B$ in the following way: for each vertex $v \in B_m$, select $\lceil \alpha d_v \rceil - m$ vertices from $N(v)$ that are not in $A$ and add them to $B$. Consider the set $D = A \cup B$. It is easy to see that $D$ is an $\alpha$-rate dominating set. The expectation of $|D|$ is:

$$E(|D|) \leq E(|A|) + E(|B|) \leq \sum_{i=1}^{n} P(v_i \in A) + \sum_{i=1}^{n} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} (\lceil \alpha d_i \rceil - m) P(v_i \in B_m)$$

$$= pn + \sum_{i=1}^{n} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} (\lceil \alpha d_i \rceil - m) \left( \begin{array}{c} d_i + 1 \\ m \end{array} \right) p^m (1 - p)^{d_i + 1 - m}$$

$$\leq pn + \sum_{i=1}^{n} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} \left( \begin{array}{c} d_i + 1 \\ \lceil \alpha d_i \rceil - 1 \end{array} \right) \left( \begin{array}{c} \lceil \alpha d_i \rceil - 1 \\ m \end{array} \right) p^m (1 - p)^{d_i + 1 - m}$$

$$= pn + \sum_{i=1}^{n} \left( \begin{array}{c} d_i + 1 \\ \lceil \alpha d_i \rceil - 1 \end{array} \right) (1 - p)^{d_i - \lceil \alpha d_i \rceil + 2} \times$$

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3.3 New Upper Bounds for $\alpha$-Rate Domination

\[
\times \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} \left( \begin{array}{c} \lceil \alpha d_i \rceil - 1 \\ m \end{array} \right) p^m (1 - p)^{\lceil \alpha d_i \rceil - 1 - m}
\]

\[
= pn + \sum_{i=1}^{n} \left( \begin{array}{c} d_i + 1 \\ \lceil \alpha d_i \rceil - 1 \end{array} \right) (1 - p)^{d_i - \lceil \alpha d_i \rceil + 2}
\]

\[
\leq pn + (1 - p)^{\lceil \delta (1 - \alpha) \rceil + 2} \sum_{i=1}^{n} \left( \begin{array}{c} d_i + 1 \\ \lceil \alpha d_i \rceil - 1 \end{array} \right)
\]

\[
= pn + (1 - p)^{\hat{\delta} + 1} \tilde{d}_\alpha n,
\]

since

\[
(\lceil \alpha d_i \rceil - m) \left( \begin{array}{c} d_i + 1 \\ m \end{array} \right) \leq \left( \begin{array}{c} d_i + 1 \\ \lceil \alpha d_i \rceil - 1 \end{array} \right) \left( \begin{array}{c} \lceil \alpha d_i \rceil - 1 \\ m \end{array} \right).
\]

Thus,

\[
\mathbb{E}(|D|) \leq pn + (1 - p)^{\hat{\delta} + 1} \tilde{d}_\alpha n. \tag{3.8}
\]

Minimizing the expression (3.8) with respect to $p$, we obtain

\[
\mathbb{E}(|D|) \leq \left( 1 - \frac{\hat{\delta}}{(1 + \frac{\hat{\delta}}{\delta + 1} \tilde{d}_\alpha)} \right) n,
\]

as required. The proof of Theorem 33 is complete.

\[\square\]

**Corollary 6** For any graph $G$,

\[
\gamma_{\times \alpha}(G) \leq \frac{\ln(\hat{\delta} + 1) + \ln \tilde{d}_\alpha + 1}{\hat{\delta} + 1} n. \tag{3.9}
\]

**Proof:** Using an approach similar to that in the proof of Corollary 5, the result follows if we put

\[
p = \min \left\{ 1, \frac{\ln(\hat{\delta} + 1) + \ln \tilde{d}_\alpha}{\hat{\delta} + 1} \right\}
\]
and use the inequality \(1 - p \leq e^{-p}\) to estimate the expression (3.8) as follows:

\[
E(|D|) \leq pn + e^{-p(\delta+1)d_\alpha n}.
\]

Note that, similar to Corollary 5, the bound of Corollary 6 also generalizes the classical upper bound (1.1). However, the probabilistic construction used to obtain the bounds (3.7) and (3.9) is different from that to obtain the bounds (3.3) and (3.6).

3.4 Effective Randomized Domination

In this section, we present randomized algorithms for constructing \(\alpha\)-dominating and \(\alpha\)-rate dominating sets, whose expected sizes satisfy the upper bounds of Theorems 32 and 33. The algorithms follow from probabilistic constructions used to prove the corresponding theorems.

3.4.1 \(\alpha\)-Domination

Algorithm 3 is a randomized algorithm to find an \(\alpha\)-dominating set \(D\), whose size satisfies the upper bound of Theorem 32 with a positive probability. In other words, the expectation of the size of the set \(D\) returned by Algorithm 3 satisfies the upper bound of Theorem 32.
3.4 Effective Randomized Domination

Algorithm 3: Randomized $\alpha$-dominating set

- **Input**: A graph $G$ and a real number $\alpha$, $0 < \alpha \leq 1$.
- **Output**: An $\alpha$-dominating set $D$ of $G$.

```
begin
    Compute $p = 1 - 1/(1 + \hat{\delta})^{\frac{1}{\hat{\delta}}}$;
    Initialize $A = \emptyset$; /* Form a set $A \subseteq V(G)$ */
    foreach vertex $v \in V(G)$ do
        with the probability $p$, decide if $v \in A$ or $v \notin A$;
    end
    Initialize $B = \emptyset$; /* Form a set $B \subseteq V(G) - A$ */
    foreach vertex $v \in V(G) - A$ do
        if $|N(v) \cap A| < \alpha d_v$ then
            /* $v$ is dominated by fewer than $\alpha d_v$ vertices of $A$ */
            add $v$ into $B$;
        end
    end
    Put $D = A \cup B$; /* $D$ is a $\alpha$-dominating set */
    return $D$;
end
```

3.4.2 $\alpha$-Rate Domination

The following Algorithm 4 is a randomized algorithm to find an $\alpha$-rate dominating set $D$. The expectation of the size of the $\alpha$-rate dominating set $D$ returned by Algorithm 4 satisfies the upper bound of Theorem 33.
3.4 Effective Randomized Domination

Algorithm 4: Randomized $\alpha$-rate dominating set

Input: A graph $G$ and a real number $\alpha$, $0 < \alpha \leq 1$.
Output: An $\alpha$-rate dominating set $D$ of $G$.

begin
Compute $p = 1 - 1 / \left( (1 + \hat{\delta}) \hat{d}_\alpha \right)^{1/\hat{\delta}}$;
Initialize $A = \emptyset$; /* Form a set $A \subseteq V(G)$ */
foreach vertex $v \in V(G)$ do
  with the probability $p$, decide if $v \in A$ or $v \notin A$;
end
Initialize $B = \emptyset$; /* Form a set $B \subseteq V(G) - A$ */
foreach vertex $v \in V(G)$ do
  Compute $r = |N[v] \cap A|$;
  if $r < \alpha d_v$ then
    add any $\lceil \alpha d_v \rceil - r$ vertices from $N[v] - A$ into $B$;
  end
end
Put $D = A \cup B$; /* $D$ is an $\alpha$-rate dominating set */
return $D$;
end

3.4.3 Complexity and Implementation

For Algorithm 3, a complexity analysis similar to that of Algorithm 1 shows that it can take up to $O(m)$ steps to find an $\alpha$-dominating set. Similarly, Algorithm 4 takes up to $O(m)$ time to find an $\alpha$-rate dominating set, i.e. it is linear in the number of edges, which, in general, can be quadratic in the number of vertices.

Algorithms 3 and 4 are presented here in a form consistent with the proofs of the corresponding theorems. But it is possible to get a smaller $\alpha$- or $\alpha$-rate dominating set $D$ by constructing the output sets $D$ more efficiently and effectively, using a recursive extension of the corresponding initial set $A$, as it is discussed in the corresponding subsection of Chapter 2.
It is easy to see that, as soon as the probability $p$ is known to all the vertices, the presented algorithms can be easily and efficiently implemented in parallel or as local distributed algorithms. This is particularly important in case of wireless sensor networks (WSN), see [66] for details. To compute the probability $p$ and to distribute its value to all the network nodes (graph vertices) in a WSN, one needs to use data gathering and data distribution rounds coordinated from a base station or a selected super-node in the WSN. When this is done, to construct the corresponding multiple dominating set for the whole network (graph), each network node (graph vertex) only needs to gather and communicate information locally in its own neighbourhood.

3.5 Final Remarks and Open Problems

To the best of our knowledge, the concept of $\alpha$-domination is still to be explored in WSNs. Construction and analysis of multiple dominating sets should lead to a better balance between efficiency and fault tolerance in WSNs and help to extend the functional lifetime of a network. It seems reasonable to do simulations with random data by analogy with the models and results presented in [27].

Notice that the concept of the $\alpha$-rate domination number $\gamma_{\times,\alpha}(G)$ is ‘opposite’ to the $\alpha$-independent $\alpha$-domination number $i_{\alpha}(G)$ as defined in [26]. A set $X \subseteq V(G)$ is defined to be an $\alpha$-independent dominating set of $G$ if for any vertex $v \in V(G)$,

$$|N[v] \cap X| \leq \alpha d_v.$$ 

The minimum cardinality of an $\alpha$-independent dominating set of $G$ is called the $\alpha$-independent domination number $i_{\alpha}(G)$. 
3.5 Final Remarks and Open Problems

It would also be interesting to use a probabilistic construction to obtain an upper bound for $i_{\alpha}(G)$ and to find a randomized algorithm to construct $\alpha$-independent $\alpha$-dominating sets of ‘small sizes’.

We wonder if it is possible to derandomize algorithms presented here or to obtain independent deterministic algorithms to find corresponding dominating sets satisfying the upper bounds of Theorems 32 and 33. Algorithms approximating the $\alpha$- and $\alpha$-rate domination numbers up to a certain degree of precision would be interesting as well.
Chapter 4

On Roman, Global and Restrained Domination in Graphs

4.1 Introduction

Let $\mathcal{H}$ be a $k$-uniform hypergraph with $n$ vertices and $m$ hyperedges. A $k$-uniform hypergraph is a hypergraph, in which the cardinality for all hyperedges is equal to $k$. The transversal number $\tau(\mathcal{H})$ of $\mathcal{H}$ is the minimum cardinality of a set of vertices that intersects all edges of $\mathcal{H}$. Alon \[2\] proved a fundamental result that if $k > 1$, then

$$\tau(\mathcal{H}) \leq \frac{\ln k}{k}(n + m).$$
He also showed that this bound is asymptotically best possible, i.e. there exists a $k$-uniform hypergraph $\mathcal{H}$ such that for sufficiently large $k$,

$$\tau(\mathcal{H}) = \frac{\ln k}{k}(n + m)(1 + o(1)).$$

Alon \cite{2} gives an interesting probabilistic construction of such a hypergraph $\mathcal{H}$. In fact, $\mathcal{H}$ is a random $k$-uniform hypergraph on $[k \ln k]$ vertices with $k$ edges constructed by choosing each edge randomly and independently according to a uniform distribution on $k$-subsets of the vertex set. This construction easily implies that the bound for the domination number in (1.1) (see Theorem 13) is asymptotically best possible:

**Theorem 34** \cite{2}  \textit{When $n$ is large there exists a graph $G$ such that}

$$\gamma(G) \geq \frac{\ln(\delta + 1) + 1}{\delta + 1} n(1 + o(1)).$$

In this chapter, new upper bounds for the global domination and Roman domination numbers are presented. Using a modification of Alon’s construction, we also prove that these results are asymptotically best possible. Moreover, in this chapter upper bounds for the restrained domination and total restrained domination numbers for large classes of graphs will be given, and it is shown that, for almost all graphs, the restrained domination number is equal to the domination number, and the total restrained domination number is equal to the total domination number. A number of open problems are posed.
4.1.1 Global Domination

The concept of global domination was introduced by Brigham and Dutton [15] and also by Sampathkumar [75]. It is a variant of the domination number. A set $X$ is called a global dominating set if $X$ is a dominating set in both $G$ and its complement $\overline{G}$. The minimum cardinality of a global dominating set of $G$ is called the global domination number $\gamma_g(G)$. There are a number of bounds on the global domination number $\gamma_g(G)$. Brigham and Dutton [15] give the following bounds for the global domination number in terms of order, minimum and maximum degrees, and the domination number of $G$:

**Theorem 35 ([15])** If either $G$ or $\overline{G}$ is disconnected, then

$$\gamma_g(G) = \max\{\gamma(G), \gamma(\overline{G})\}.$$  

**Theorem 36 ([15])** For any graph $G$, either

$$\gamma_g(G) = \max\{\gamma(G), \gamma(\overline{G})\}$$

or

$$\gamma_g(G) \leq \min\{\Delta(G), \Delta(\overline{G})\} + 1.$$ 

**Theorem 37 ([15])** For any graph $G$, if $\delta(G) = \delta(\overline{G}) \leq 2$, then

$$\gamma_g(G) \leq \delta(G) + 2;$$ otherwise

$$\gamma_g(G) \leq \max\{\delta(G), \delta(\overline{G})\} + 1.$$
4.1.2 Roman Domination

Another variant of the domination number, the Roman domination number, was introduced by Stewart [80]. In [80] and [73], a Roman dominating function (RDF) of a graph $G$ is defined as a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of an RDF is defined as the value

$$f(V(G)) = \sum_{v \in V(G)} f(v).$$

The Roman domination number of a graph $G$, denoted $\gamma_R(G)$, is equal to the minimum weight of an RDF on $G$. In fact, Roman domination is of both historical and mathematical interest. Emperor Constantine had the requirement that an army or legion could be sent from its home to defend a neighbouring location only if there was a second army which would stay and protect the home. Thus, there were two types of armies: stationary and travelling. Each vertex with no army must have a neighbouring vertex with a travelling army. Stationary armies then dominate their own vertices, and a vertex with two armies is dominated by its stationary army, and its open neighbourhood is dominated by the travelling army. Thus, the definition of Roman domination has its historical background and it can be used for the problems of this type, which arise in military and commercial decision making. The following results about the Roman domination number are known:

**Theorem 38 ([22])** For any graph $G$,

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G).$$
4.1 Introduction

Theorem 39 ([22]) For any graph $G$ of order $n$ and maximum degree $\Delta$,

$$\gamma_R(G) \geq \frac{2n}{\Delta+1}.$$ 

4.1.3 Restrained Domination

Telle and Proskurowski [82] introduced restrained domination as a vertex partitioning problem. A dominating set $X$ of a graph $G$ is called a restrained dominating set if every vertex in $V(G) - X$ is adjacent to a vertex in $V(G) - X$. If, in addition, every vertex of $X$ is adjacent to a vertex of $X$, then $X$ is called a total restrained dominating set. The minimum cardinality of a restrained dominating set of $G$ is the restrained domination number $\gamma_r(G)$, and the minimum cardinality of a total restrained dominating set of $G$ is the total restrained domination number $\gamma_{tr}(G)$. For these parameters, the following upper bounds have been found:

Theorem 40 ([28]) If $\delta \geq 2$, then

$$\gamma_r(G) \leq n - \Delta.$$ 

Theorem 41 ([50]) If $G$ is a connected graph with $n \geq 4$, $\delta \geq 2$ and $\Delta \leq n - 2$, then

$$\gamma_{tr}(G) \leq n - \frac{\Delta}{2} - 1.$$
4.2 New Upper Bounds for the Global Domination Number

The following theorem provides an upper bound for the global domination number. In what follows, we denote $\bar{\delta} = \delta(G)$ and $\delta' = \min\{\delta, \bar{\delta}\}$.

**Theorem 4.2** For any graph $G$ with $\delta' > 0$,

$$\gamma_g(G) \leq \left(1 - \frac{\delta'}{2^{1/\delta'} (1 + \delta')^{1 + 1/\delta'}}\right)n.$$

**Proof:** Let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with the probability

$$p = 1 - \frac{1}{2^{1/\delta'} (1 + \delta')^{1/\delta'}}.$$

Let us denote $B = V(G) - N[A]$ and $C = \{v_i \in V(G), \ v_i \text{ is not dominated by } A \text{ in } G\}$.

It is easy to show that

$$P[v_i \in B] = (1 - p)^{1 + \deg(v_i)} \leq (1 - p)^{1 + \delta}$$
4.2 New Upper Bounds for the Global Domination Number

It is obvious that the set \( D = A \cup B \cup C \) is a global dominating set. The expectation of \(|D|\) is

\[
\mathbb{E}[|D|] \leq \mathbb{E}[|A|] + \mathbb{E}[|B|] + \mathbb{E}[|C|]
\]

\[
= pn + \sum_{i=1}^{n} \mathbb{P}[v_i \in B] + \sum_{i=1}^{n} \mathbb{P}[v_i \in C]
\]

\[
\leq pn + (1 - p)^{1+\delta} n + (1 - p)^{1+\delta} n
\]

\[
\leq pn + 2(1 - p)^{1+\min\{\delta, \delta\}} n
\]

\[
= pn + 2(1 - p)^{1+\delta'} n
\]

\[
= \left( 1 - \frac{\delta'}{2^{1/\delta'} (1 + \delta')^{1+1/\delta'}} \right) n,
\]

as required. The proof of the theorem is complete.

The proof of Theorem 42 implies the following upper bound, which is asymptotically same as the bound of Theorem 42.

**Corollary 7** For any graph \( G \),

\[
\gamma_g(G) \leq \frac{\ln(\delta' + 1) + \ln 2 + 1}{\delta' + 1} n.
\]
4.2 New Upper Bounds for the Global Domination Number

Proof: Using the inequality $1 - p \leq e^{-p}$, we obtain the following estimation of the expression (4.1):

$$E[|D|] \leq pn + 2e^{-p(\delta' + 1)}n.$$ 

If we put $p = \min\{1, \frac{\ln(\delta' + 1) + \ln 2}{\delta' + 1}\}$, then

$$E[|D|] \leq \frac{\ln(\delta' + 1) + \ln 2 + 1}{\delta' + 1} n,$$

as required.

We now prove that the upper bound of Corollary 7 and therefore of Theorem 42 is asymptotically best possible.

**Theorem 43** When $n$ is large, there exists a graph $G$ such that

$$\gamma_g(G) \geq \frac{\ln(\delta' + 1) + \ln 2 + 1}{\delta' + 1} n(1 + o(1)).$$

Proof: Let us modify Alon’s probabilistic construction described in the introduction as follows. Let $F$ be a complete graph $K_{[\delta \ln \delta]}$, and let us denote $F = V(F)$. Next, we add a set of new vertices $V = \{v_1, \ldots, v_\delta\}$, where each vertex $v_i$ is adjacent to $\delta$ vertices that are randomly chosen from the set $F$. Let us add a new component $K_{\delta + 1}$ and denote the resulting graph by $G$, which has $n = [\delta \ln \delta] + 2\delta + 1$ vertices. Note that $\delta' = \delta$ because $\bar{\delta} > \delta$. We will prove that with high probability

$$\gamma_g(G) \geq \frac{\ln \delta'}{\delta'} n(1 + o(\delta'))$$

$$= \frac{\ln \delta}{\delta} n(1 + o(1))$$

$$= \ln^2 \delta(1 + o(\delta)).$$
4.2 New Upper Bounds for the Global Domination Number

Let us denote by $H$ the graph $G$ without the component $K_{\delta+1}$. It is obvious that

$$\gamma_g(G) = \gamma(H) + 1.$$ 

Therefore, the result will follow if we can prove that with high probability

$$\gamma(H) > \ln^2 \delta (1 + o_\delta(1)).$$

Without loss of generality we may only consider dominating sets in $H$ that are subsets of $F$. Let us consider a dominating set $X$ in $H$ such that $X \subseteq F$ and $|X| \leq \ln^2 \delta - \ln \delta \ln \ln^5 \delta$. It is easy to show that the probability of the set $X$ not dominating a vertex $v_i \in V$ is

$$P[X \text{ does not dominate } v_i] = \left( \frac{|F| - |X|}{\delta} \right)^{|F|/\delta} \geq \left( \frac{|F| - |X| - \delta}{|F| - \delta} \right)^\delta = \left( 1 - \frac{|X|}{|F| - \delta} \right)^\delta.$$

Using the inequality $1 - x \geq e^{-x}(1 - x^2)$ if $x < 1$, we obtain the following estimation:

$$P[X \text{ does not dominate } v_i] \geq e^{-\frac{\ln^2 \delta - \ln \delta \ln \ln^5 \delta}{\delta} \ln \ln^5 \delta} \left( 1 - \left( \frac{\ln^2 \delta - \ln \delta \ln \ln^5 \delta}{\delta \ln \delta - \delta} \right)^2 \right)^\delta = e^{-\frac{\ln \delta + \ln \ln^5 \delta}{\ln \delta - \delta} \ln \ln^5 \delta} (1 + o_\delta(1)) = e^{\ln (\frac{\ln^5 \delta}{\delta}) (1 + o_\delta(1))} (1 + o_\delta(1))$$

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4.2 New Upper Bounds for the Global Domination Number

\[ = \left( \frac{\ln^5 \delta}{\delta} \right)^{1+o_\delta(1)} \left( 1 + o_\delta(1) \right) \]
\[ \geq \frac{\ln^4 \delta}{\delta}. \]

Thus, we conclude that

\[ P[X \text{ dominates } V] \leq \left( 1 - \frac{\ln^4 \delta}{\delta} \right)^{\delta} \]
\[ \leq e^{-\ln^4 \delta}. \]

It is obvious that the number of choices for the set \( X \) is less than \( \sum_{i=0}^{\ln^2 \delta} \binom{|F|}{i} \).

We have

\[ \sum_{i=0}^{\ln^2 \delta} \binom{|F|}{i} < \ln^2 \delta \binom{\delta \ln \delta}{\ln^2 \delta} \]
\[ < (\delta \ln \delta)^{\ln^2 \delta} \]
\[ < e^{2 \ln^3 \delta}. \]

Now we can estimate the probability that the domination number of the graph \( H \) is less than or equal to \( \ln^2 \delta - \ln \delta \ln \ln^5 \delta \):

\[ P[\gamma(H) \leq \ln^2 \delta - \ln \delta \ln \ln^5 \delta] < \sum_{i=0}^{\ln^2 \delta} \binom{|F|}{i} P[X \text{ dominates } V] \]
\[ < e^{2 \ln^3 \delta - \ln^3 \delta} \]
\[ = o_\delta(1). \]
Therefore, with high probability

\[
\gamma(H) > \ln^2 \delta - \ln \delta \ln \ln^5 \delta \\
= \ln^2 \delta (1 + o_\delta(1)),
\]

as required. The proof of the theorem is complete.  \( \square \)

### 4.3 New Upper Bounds for the Roman Domination Number

The following theorem provides an upper bound for the Roman domination number:

**Theorem 44** For any graph \( G \) with \( \delta > 0 \),

\[
\gamma_R(G) \leq 2 \left( 1 - \frac{2^{1/\delta}}{(1 + \delta)^{1/\delta}} \right)^n.
\]

**Proof:** Let \( A \) be a set formed by an independent choice of vertices of \( G \), where each vertex is selected with the probability

\[
p = 1 - \left( \frac{2}{1 + \delta} \right)^{1/\delta}.
\]

We denote \( B = N[A] - A \) and \( C = V(G) - N[A] \). Let us assume that \( f \) is a function \( f : V(G) \to \{0, 1, 2\} \) and assign \( f(v_i) = 2 \) for each \( v_i \in A \), \( f(v_i) = 0 \) for each \( v_i \in B \) and \( f(v_i) = 1 \) for each \( v_i \in C \). It is obvious that \( f \) is a Roman
4.3 New Upper Bounds for the Roman Domination Number

dominating function and

\[ f(V(G)) = 2|A| + |C| \]

It is easy to show that

\[ P[v_i \in C] = (1 - p)^{1 + \deg(v_i)} \leq (1 - p)^{1 + \delta}. \]

The expectation of \( f(V(G)) \) is

\[
E[f(V(G))] \leq 2E[|A|] + E[|C|] \\
= 2pn + \sum_{i=1}^{n} P[v_i \in C] \\
\leq 2pn + (1 - p)^{1 + \delta} n \\
= 2 \left( 1 - \frac{\delta^{2/\delta}}{(1 + \delta)^{1+1/\delta}} \right) n. \tag{4.2}
\]

Since the expectation is an average value, there exists a particular Roman dominating function of the above weight, as required. The proof of the theorem is complete.

Theorem 44 implies the following upper bound.

**Corollary 8** For any graph \( G \) with \( \delta > 0 \),

\[ \gamma_R(G) \leq \frac{2 \ln(\delta + 1) - \ln 4 + 2}{\delta + 1} n. \]
4.3 New Upper Bounds for the Roman Domination Number

Proof: Using the inequality $1 - p \leq e^{-p}$, we obtain the following estimation of the expression (4.2):

$$E[f(V(G))] \leq 2pn + e^{-p(\delta+1)n}.$$

If we put $p = \frac{\ln(\delta+1) - \ln 2}{\delta+1}$, then

$$E[f(V(G))] \leq \frac{2\ln(\delta+1) - \ln 4 + 2}{\delta+1}n,$$

as required.

Note that the result of Corollary 8 was also proved in [22], even though the upper bound in [22] contains a misprint.

Now let us prove that the upper bound of Corollary 8 and therefore of Theorem 44 is asymptotically best possible.

Theorem 45 When $n$ is large, there exists a graph $G$ such that

$$\gamma_R(G) \geq \frac{2\ln(\delta+1) - \ln 4 + 2}{\delta+1}n(1 + o(1)).$$

Proof: Let $F$ be a complete graph $K_{[\delta \ln \delta]}$, and let us denote $F = V(F)$. Next, we add a set of new vertices $V = \{v_1, ..., v_\delta\}$, where each vertex $v_i$ is adjacent to $\delta$ vertices that are randomly chosen from the set $F$. The resulting graph is denoted by $G$ and it has $n = [\delta \ln \delta] + \delta$ vertices. We will prove that with positive probability

$$\gamma_R(G) \geq \frac{2\ln \delta}{\delta}n(1 + o_\delta(1))$$
4.3 New Upper Bounds for the Roman Domination Number

\[ \gamma_R(G) = 2 \ln^2 \delta (1 + o(\delta)). \]

Let \( f = (D_0, D_1, D_2) \) be a \( \gamma_R \)-function of \( G \), i.e. \( f \) is a Roman dominating function and \( f(V(G)) = \gamma_R(G) \). It is easy to see that we may assume that \( D_2 \subseteq F \) and \( D_1 \subseteq V \).

Let us consider two cases. If \( |D_2| > \ln^2 \delta - \ln \delta \ln \ln^4 \delta \), then

\[ f(V(G)) > 2 \ln^2 \delta (1 + o(\delta)), \]

as required. If \( |D_2| \leq \ln^2 \delta - \ln \delta \ln \ln^4 \delta \), then, similar to the proof of Theorem 4.3, we can prove that the probability of the set \( D_2 \) dominating a vertex \( v_i \in V \) is

\[ P[D_2 \text{ dominates } v_i] \leq 1 - \frac{\ln^2 \delta}{\delta}. \]

Let us consider the random variable \( |N(D_2) \cap V| \). The expectation of \( |N(D_2) \cap V| \) is

\[ E[|N(D_2) \cap V|] = \sum_{i=1}^{\delta} P[D_2 \text{ dominates } v_i] \leq \delta - \ln^3 \delta. \]

Thus, we can conclude that there exists a graph \( G \), for which \( |D_1| \geq \ln^3 \delta \), i.e.

\[ f(V(G)) \geq \ln^3 \delta \]
\[ > 2 \ln^2 \delta (1 + o(\delta)), \]

as required.
4.4 Results for Restrained and Total Restrained Domination

Theorem 4.3 implies that when $\delta(G)$ is large, $\gamma(G)/n$ is close to 0 for any graph $G$. Similar results were proved for the global and Roman domination numbers in the previous sections. However, for the total restrained domination numbers this is not the case, because for any $\delta$ there exists (see [50]) an infinite family of graphs $G$ with minimum degree $\delta$, for which $\gamma_{tr}(G)/n \to 1$ when $n$ tends to $\infty$. The above is also true for the restrained domination number. Thus, for the class of all graphs, it is impossible to provide an upper bound for these parameters similar to the result of Theorem 4.3. In this section, we will give such upper bounds for large classes of graphs.

Let us first find the restrained domination number of a ‘typical’ graph. Let $0 < p < 1$ be fixed and put $q = 1 - p$. Denote by $\mathcal{G}(n, P[\text{edge}] = p)$ the discrete probability space consisting of all graphs with $n$ fixed and labelled vertices, in which the probability of each graph with $M$ edges is $p^M q^{N-M}$, where $N = \binom{n}{2}$. Equivalently, the edges of a labelled random graph are chosen independently and with the same probability $p$. We say that a random graph $G$ satisfies a property $Q$ if

$$\mathbb{P}[G \text{ has } Q] \to 1 \text{ as } n \to \infty.$$ 

If a random graph $G$ has a property $Q$, then we also say that almost all graphs satisfy $Q$.

It turns out that, for almost all graphs, the restrained domination number is equal to the domination number, which has two points of concentration, and
4.4 Results for Restrained and Total Restrained Domination

the total restrained domination number is equal to the total domination number. This is formulated in the following theorem, which is based on the fundamental results of Bollobás [14] and Weber [85]. Remind that a dominating set $X$ is called a total dominating set if every vertex of $X$ is adjacent to a vertex of $X$. The total domination number $\gamma_t(G)$, which is one of the basic domination parameters, is the minimum cardinality of a total dominating set of $G$.

**Theorem 46** For almost every graph, $\gamma_r(G) = \gamma(G)$ and $\gamma_{tr}(G) = \gamma_t(G)$. Moreover,

$$\gamma_r(G) = \lfloor \log n - 2 \log \log n + \log \log e \rfloor + \epsilon,$$

where $\epsilon = 1$ or $2$, and $\log$ denotes the logarithm with base $1/q$.

**Proof:** Bollobás [14] proved that a random graph $G$ satisfies

$$| \delta(G) - pn + (2pqn \log n)^{1/2} - \left( \frac{pqn}{8 \log n} \right)^{1/2} \log \log n | \leq C(n) \left( \frac{n}{\log n} \right)^{1/2},$$

where $C(n) \to \infty$ arbitrarily slowly. Therefore,

$$\delta(G) = pn(1 + o(1)).$$

Weber [85] showed that the domination number of a random graph $G$ is equal to

$$k + 1 \text{ or } k + 2,$$

where

$$k = \lfloor \log n - 2 \log \log n + \log \log e \rfloor$$
4.4 Results for Restrained and Total Restrained Domination

and log denotes the logarithm with base $1/q$. Let us consider a minimum dominating set $D$ of this size. We have

$$|D| = \log n(1 + o(1)).$$

For any vertex $v \in V(G) - D$ and large $n$,

$$\deg v \geq \delta = pn(1 + o(1)) > \log n(1 + o(1)) = |D|,$$

since $p$ is fixed. Therefore, the vertex $v$ is adjacent to a vertex in $V(G) - D$, i.e. $D$ is a restrained dominating set.

Now let us consider a minimum total dominating set $T$, i.e. $|T| = \gamma_t(G)$. It is not difficult to see that

$$\gamma_t(G) \leq 2\gamma(G).$$

Therefore,

$$|T| \leq 2|D| = 2\log n(1 + o(1)).$$

Thus, for any vertex $v \in V(G) - T$ and large $n$,

$$\deg v \geq \delta = pn(1 + o(1)) > 2\log n(1 + o(1)) \geq |T|,$$

since $p$ is fixed. Therefore, the vertex $v$ is adjacent to a vertex in $V(G) - T$, i.e. $T$ is a total restrained dominating set, which is also minimum. The result follows.

However, the property of a ‘typical’ graph stated in the above theorem cannot
be used as a bound for the (total) restrained domination number for a given graph. Let us find such upper bounds for large classes of graphs.

**Proposition 1** If \( \delta > 0 \) and \( n < \delta^2 / (\ln \delta + 1) \), then

\[
\gamma_r(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n
\]

and

\[
\gamma_{tr}(G) \leq \frac{\ln \delta + 1}{\delta} n.
\]

**Proof:** Using Theorem 13, let us consider a dominating set \( D \) such that

\[
|D| \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n.
\]

Note that the condition \( n < \delta^2 / (\ln \delta + 1) \) can be written as follows:

\[
\delta > \frac{\ln \delta + 1}{\delta} n.
\]

Now, for any vertex \( v \in V(G) - D \),

\[
\deg v \geq \delta > \frac{\ln(\delta + 1)}{\delta} n > \frac{\ln(\delta + 1) + 1}{\delta + 1} n \geq |D|.
\]

Therefore, the vertex \( v \) is adjacent to a vertex in \( V(G) - D \), i.e. \( D \) is a restrained dominating set.

Using the probabilistic method of the proof of Theorem 13, we can show that for any graph \( G \) with \( \delta > 0 \),

\[
\gamma_t(G) \leq \frac{\ln \delta + 1}{\delta} n.
\]
Let us consider a total dominating set $T$ such that

$$|T| \leq \frac{\ln \delta + 1}{\delta} n.$$  

For any vertex $v \in V(G) - T$,

$$\deg v \geq \delta > \frac{\ln \delta + 1}{\delta} n \geq |T|.$$  

Therefore, the vertex $v$ is adjacent to a vertex in $V(G) - T$, i.e. $T$ is a total restrained dominating set.

Note that the result of Bollobás [13] on the minimum degree implies that the condition $n < \frac{\delta^2}{\ln \delta + 1}$ is satisfied for almost all graphs, i.e. Proposition 1 gives upper bounds for a very large class of graphs. Moreover, in the class of graphs with $n < \frac{\delta^2}{\ln \delta + 1}$, the upper bounds of Proposition 1 cannot be improved. This can be proved in the same way as Theorem 34.

The matching number of a graph $G$, denoted by $\beta_1(G)$, is the largest number of pairwise non-adjacent edges in $G$. This number is also called the edge independence number.

**Theorem 47** For any graph $G$ with $\delta > 0$,

$$\gamma_r(G) \leq \frac{2 \ln(\delta + 1) + \delta + 3}{\delta + 1} n - 2\beta_1$$

and

$$\gamma_{tr}(G) \leq \frac{2 \ln \delta + \delta + 2}{\delta} n - 2\beta_1.$$  

**Proof:** Let us consider a minimum dominating set $|D|$ of the graph $G$, i.e.
4.4 Results for Restrained and Total Restrained Domination

$|D| = \gamma(G)$. Let $M$ be a matching with $\beta_1(G)$ edges:

$$M = (e_1, e_2, ..., e_{\beta_1}).$$

Without loss of generality we may assume that the first $k$ edges of $M$ have at least one end in $D$, thus $\beta_1 - k$ edges of $M$ have both ends in $V(G) - D$. It is obvious that

$$k \leq |D| = \gamma(G).$$

Therefore, at least $\beta_1(G) - \gamma(G)$ edges in $M$ have both end vertices in $V(G) - D$. Note that $\beta_1(G) \geq \gamma(G)$, because each vertex of a total dominating set $S$ has a private neighbour not in $S$, thus providing a matching of size $|S|$, which is at least $\gamma(G)$.

Now we form a restrained dominating set $D'$ by adding to $D$ all vertices not belonging to the last $\beta_1 - k$ edges of $M$. We obtain

$$\gamma_r(G) \leq |D'| = n - 2(\beta_1 - k) \leq n - 2\beta_1 + 2\gamma.$$ 

By Theorem 13,

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n.$$ 

Therefore,

$$\gamma_r(G) \leq \frac{2\ln(\delta + 1) + \delta + 3}{\delta + 1} n - 2\beta_1,$$

as required.

Let us prove the latter upper bound. Consider a minimum total dominating set $T$ and the above matching $M$. Using a similar technique, we can construct a
4.4 Results for Restrained and Total Restrained Domination

total restrained dominating set $T'$ such that

$$\gamma_{tr}(G) \leq |T'| \leq n - 2\beta_1 + 2\gamma_t.$$ 

Using the probabilistic method of the proof of Theorem 13, we can show that for any graph $G$ with $\delta > 0$,

$$\gamma_t(G) \leq \frac{\ln \delta + 1}{\delta} n.$$ 

Therefore,

$$\gamma_{tr}(G) \leq \frac{2\ln \delta + \delta + 2}{\delta} n - 2\beta_1,$$

as required.

A matching is called **perfect** if it contains all vertices of a graph (or all vertices but one if $n$ is odd). The following corollary follows immediately from the above theorem:

**Corollary 9** If $G$ has a perfect matching, then

$$\gamma_r(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} 2n + \epsilon$$

and

$$\gamma_{tr}(G) \leq \frac{\ln \delta + 1}{\delta} 2n + \epsilon,$$

where $\epsilon = 0$ if $n$ is even and $\epsilon = 1$ otherwise.

It may be pointed out that the class of graphs with a perfect matching includes all Hamiltonian graphs. It is well known that almost all graphs are Hamiltonian

[65], thus Corollary 9 provides upper bounds for a very large class of graphs.
4.5 Concluding Remarks and Open Problems

In this section, a number of open problems on the domination parameters discussed in this chapter are formulated. Each of these problems can be considered a subject for future research.

By Theorem 46, the total restrained domination number is equal to the total domination number for almost every graph. However, we do not know points of concentration of the total domination number for almost all graphs. But for the domination number such a result is already known [85]. The problem can be formulated as follows:

**Problem 1** For almost all graphs, find points of concentration of the total, global and Roman domination numbers.

Theorem 13 is formulated for all graphs and it gives an excellent upper bound if $\delta$ is big. However, for small values of $\delta$, better (sharp) bounds are known, see Theorems 16, 17 and 18. This is also true for many upper bounds proved in this chapter. The new upper bounds are good when $\delta$ is not small. The question whether there can be found better upper bounds for small values of $\delta$ arises. Therefore, the following problems are formulated:

**Problem 2** Determine sharp upper bounds for the global and Roman domination numbers of a graph with small minimum degree.

**Problem 3** Determine sharp upper bounds for the restrained and total restrained domination numbers of a graph with a perfect matching and small minimum degree.
Chapter 5

Discrepancy and Signed Domination in Graphs and Hypergraphs

5.1 Introduction

For a graph $G$, a signed domination function of $G$ is a two-colouring of the vertices of $G$ with colours +1 and –1 such that the closed neighbourhood of every vertex contains more +1’s than –1’s. This concept is closely related to combinatorial discrepancy theory as shown by Füredi and Mubayi [37]. The signed domination number of $G$ is the minimum of the sum of colours for all vertices, taken over all signed domination functions of $G$. In this chapter, new upper and lower bounds for the signed domination number are presented. These new bounds improve a

\[1\] Parts of the work given in this chapter are based on a conference paper included in Appendix B.
number of known results.

5.1.1 Notation and Technical Results

The domination number can be defined equivalently by means of a dominating function, which can be considered as a characteristic function of a dominating set in $G$. A function $f : V(G) \to \{0,1\}$ is a dominating function on a graph $G$ if for each vertex $v \in V(G)$,

$$\sum_{x \in N[v]} f(x) \geq 1.$$  (5.1)

The value $\sum_{v \in V(G)} f(v)$ is called the weight $f(V(G))$ of the function $f$. It is obvious that the minimum of weights, taken over all dominating functions on $G$, is the domination number $\gamma(G)$ of $G$.

It is easy to obtain different variations of the domination number by replacing the set $\{0,1\}$ by another set of numbers. If $\{0,1\}$ is exchanged by $\{-1,1\}$, then we obtain the signed domination number. A signed domination function of a graph $G$ was defined in [29] as a function $f : V(G) \to \{-1,1\}$ such that for each $v \in V(G)$, the expression (5.1) is true. The signed domination number of a graph $G$, denoted $\gamma_s(G)$, is the minimum of weights $f(V(G))$ taken over all signed domination functions $f$ on $G$. A research on signed domination has been carried out in [29, 33, 37, 41, 46, 50, 51] and [63].

Let $d \geq 2$ be an integer and $0 \leq p \leq 1$. Let us denote

$$f(d, p) = \sum_{m=0}^{\lfloor 0.5d \rfloor} ([0.5d] - m + 1) \binom{d + 1}{m} p^m (1 - p)^{d+1-m}.$$  (5.2)

We will need the following technical results:
Lemma 1 ([37]) If $d$ is odd, then
\[
f(d + 1, p) < 2(1 - p)f(d, p).
\]

If $d$ is even, then
\[
f(d + 1, p) < \left( 2p + (1 - p) \frac{d + 4}{d + 2} \right) f(d, p).
\]

In particular, if
\[
2(1 - p)\left( 2p + (1 - p) \frac{d + 4}{d + 2} \right) < 1,
\]
then
\[
\max_{d \geq \delta} f(d, p) \in \{ f(\delta, p), f(\delta + 1, p) \}.
\]

Lemma 2 ([20]) Let $p \in [0, 1]$ and $X_1, ..., X_k$ be mutually independent random variables with
\[
P[X_i = 1 - p] = p,
P[X_i = -p] = 1 - p.
\]

If $X = X_1 + ... + X_k$ and $c > 0$, then
\[
P[X < -c] < e^{-\frac{c^2}{2pr}}.
\]

Let us also denote
\[
\tilde{d}_{0.5} = \left( \delta' + 1 \right)
\]
where
\[ \delta' = \begin{cases} \delta & \text{if } \delta \text{ is odd;} \\ \delta + 1 & \text{if } \delta \text{ is even.} \end{cases} \]

### 5.1.2 Discrepancy Theory and Signed Domination

Originated from number theory, discrepancy theory is, generally speaking, the study of irregularities of distributions in various settings. Classical combinatorial discrepancy theory is devoted to the problem of partitioning the vertex set of a hypergraph into two classes in such a way that all hyperedges are split into approximately equal parts by the classes, i.e. we are interested in measuring the deviation of an optimal partition from perfect, when all hyperedges are split into equal parts. It may be pointed out that many classical results in various areas of mathematics, e.g. geometry and number theory, can be formulated in these terms. Combinatorial discrepancy theory was introduced and studied by Beck in [6]. Also, studies on discrepancy theory have been conducted in [5, 7, 8] and [78].

Let \( \mathcal{H} = (V, E) \) be a hypergraph with the vertex set \( V \) and the hyperedge set \( E = \{E_1, ..., E_m\} \). One of the main problems in classical combinatorial discrepancy theory is to colour the elements of \( V \) by two colours in such a way that all of the hyperedges have almost the same number of elements of each colour. Such a partition of \( V \) into two classes can be represented by a function

\[ f : V \to \{+1, -1\}. \]

For a set \( E \subseteq V \), let us define the *imbalance* of \( E \) as follows:

\[ f(E) = \sum_{v \in E} f(v). \]
First defined by Beck [6], the discrepancy of $\mathcal{H}$ with respect to $f$ is

$$\mathcal{D}(\mathcal{H}, f) = \max_{E_i \in \mathcal{E}} |f(E_i)|$$

and the discrepancy of $\mathcal{H}$ is

$$\mathcal{D}(\mathcal{H}) = \min_{f: V \to \{+1, -1\}} \mathcal{D}(\mathcal{H}, f).$$

Thus, the discrepancy of a hypergraph tells us how well all its hyperedges can be partitioned. Spencer [79] proved a fundamental ‘six-standard-deviation’ result that for any hypergraph $\mathcal{H}$ with $n$ vertices and $n$ hyperedges,

$$\mathcal{D}(\mathcal{H}) \leq 6\sqrt{n}.$$ 

As shown in [3], this bound is best possible up to a constant factor. More precisely, if a Hadamard matrix of order $n > 1$ exists, then there is a hypergraph $\mathcal{H}$ with $n$ vertices and $n$ hyperedges such that

$$\mathcal{D}(\mathcal{H}) \geq 0.5\sqrt{n}.$$ 

It is well known that a Hadamard matrix of order between $n$ and $(1 - \epsilon)n$ does exist for any $\epsilon$ and sufficiently large $n$. The following important result, due to Beck and Fiala [7], is valid for a hypergraph with any number of hyperedges:

$$\mathcal{D}(\mathcal{H}) \leq 2\Delta - 1,$$
where $\Delta$ is the maximum degree of vertices of $\mathcal{H}$. They also posed the discrepancy conjecture that for some constant $K$,

$$\mathcal{D}(\mathcal{H}) < K\sqrt{\Delta}.$$ 

Another interesting aspect of discrepancy was discussed by Füredi and Mubayi in their fundamental paper [37]. A function $g : V \to \{+1, -1\}$ is called a signed domination function (SDF) of the hypergraph $\mathcal{H}$ if

$$g(E_i) = \sum_{v \in E_i} g(v) \geq 1$$

for every hyperedge $E_i \in \mathcal{E}$, i.e. each hyperedge has a positive imbalance. The signed discrepancy of $\mathcal{H}$, denoted by $SD(\mathcal{H})$, is defined in the following way:

$$SD(\mathcal{H}) = \min_{SDF_g} g(V),$$

where the minimum is taken over all signed domination functions of $\mathcal{H}$. Thus, in this version of discrepancy, the success is measured by minimizing the imbalance of the vertex set $V$, while keeping the imbalance of every hyperedge $E_i \in \mathcal{E}$ positive.

One of the main results in this context, formulated in terms of hypergraphs, is due to Füredi and Mubayi [37]:

**Theorem 48** ([37]) *Let $\mathcal{H}$ be an $n$-vertex hypergraph with hyperedge set $\mathcal{E} = \{E_1, ..., E_m\}$, and suppose that every hyperedge has at least $k$ vertices, where $k \geq 80$.*
Then
\[ \mathcal{S}\mathcal{D}(\mathcal{H}) \leq 4\sqrt{\frac{\ln k}{k}} n + \frac{1}{k} m. \]

This theorem can be easily re-formulated in terms of graphs by considering the neighbourhood hypergraph of a given graph.

**Theorem 49 ([37])** If \( G \) has \( n \) vertices and minimum degree \( \delta \geq 99 \), then
\[ \gamma_s(G) \leq \left( 4\sqrt{\frac{\ln(\delta + 1)}{\delta + 1}} + \frac{1}{\delta + 1} \right) n. \]

Moreover, Füredi and Mubayi [37] found quite good upper bounds for very small values of \( \delta \) and, using Hadamard matrices, constructed a \( \delta \)-regular graph \( G \) of order \( 4\delta \) with
\[ \gamma_s(G) \geq 0.5\sqrt{\delta} - O(1). \]

This construction shows that the upper bound in Theorem 49 is off from optimal by at most the factor of \( \sqrt{\ln \delta} \). They posed an interesting conjecture that, for some constant \( C \),
\[ \gamma_s(G) \leq \frac{C}{\sqrt{\delta}} n, \]
and proved that the above discrepancy conjecture, if true, would imply this upper bound for \( \delta \)-regular graphs. A strong result of Matoušek [63] shows that the bound is true, but the constant \( C \) in his proof is big making the result of rather theoretical interest.

The lower bound for the signed domination number given in the theorem below is formulated in terms of the degree sequence of a graph. Other lower bounds are also known, see Corollaries 13, 14 and 15.
5.2 Upper Bounds for the Signed Domination Number

**Theorem 50** ([29]) Let $G$ be a graph with degrees $d_1 \leq d_2 \leq \ldots \leq d_n$. If $k$ is the smallest integer for which

$$
\sum_{i=0}^{k-1} d_{n-i} \geq 2(n - k) + \sum_{i=1}^{n-k} d_i,
$$

then

$$
\gamma_s(G) \geq 2k - n.
$$

In this chapter, new upper and lower bounds for the signed domination number are presented. They improve the above theorems and also generalize three known results formulated in Corollaries 13, 14 and 15. Note that our results can be easily re-formulated in terms of hypergraphs. Moreover, we rectify Füredi–Mubayi’s conjecture formulated above as follows: for some $C \leq 10$ and $\alpha$, $0.18 \leq \alpha < 0.21$,

$$
\gamma_s(G) \leq \min \left\{ n \frac{Cn}{\delta^\alpha}, \sqrt{\frac{Cn}{\delta}} \right\}.
$$

(5.2)

5.2 Upper Bounds for the Signed Domination Number

The following theorem provides an upper bound for the signed domination number, which is better than the bound of Theorem 49 for ‘relatively small’ values of $\delta$. For example, if $\delta(G) = 99$, then, by Theorem 49, $\gamma_s(G) \leq 0.869n$, while Theorem 51 yields $\gamma_s(G) \leq 0.537n$. For larger values of $\delta$, the latter result is improved in Corollaries 10–12.
5.2 Upper Bounds for the Signed Domination Number

**Theorem 51** For any graph $G$ with $\delta > 1$,

$$\gamma_s(G) \leq \left(1 - \frac{2\hat{\delta}}{(1 + \hat{\delta})^{1/\hat{\delta}} d_{0.5}^{1/\hat{\delta}}}\right) n,$$

(5.3)

where $\hat{\delta} = \lfloor 0.5\delta \rfloor$.

**Proof:** Let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with the probability

$$p = 1 - \frac{1}{(1 + \hat{\delta})^{1/\hat{\delta}} d_{0.5}^{1/\hat{\delta}}}.$$

For $m \geq 0$, let us denote by $B_m$ the set of vertices $v \in V(G)$ dominated by exactly $m$ vertices of $A$ and such that $|N[v] \cap A| < \lfloor 0.5d_v \rfloor + 1$, i.e.

$$|N[v] \cap A| = m \leq \lfloor 0.5d_v \rfloor.$$

Note that each vertex $v \in V(G)$ is in at most one of the sets $B_m$ and $0 \leq m \leq \lfloor 0.5d_v \rfloor$. Then we form a set $B$ by selecting $\lfloor 0.5d_v \rfloor - m + 1$ vertices from $N[v]$ that are not in $A$ for each vertex $v \in B_m$ and adding them to $B$. We construct the set $D$ as follows: $D = A \cup B$. Let us assume that $f$ is a function $f : V(G) \rightarrow \{-1, 1\}$ such that all vertices in $D$ are labelled by 1 and all other vertices by $-1$. It is obvious that $f(V(G)) = |D| - (n - |D|)$ and $f$ is a signed domination function.

The expectation of $f(V(G))$ is

$$E[f(V(G))] = 2E[|D|] - n$$

$$= 2(E[|A|] + E[|B|]) - n$$
5.2 Upper Bounds for the Signed Domination Number

\[
\leq 2 \sum_{i=1}^{n} P(v_i \in A) + 2 \sum_{i=1}^{n} \sum_{m=0}^{\floor{0.5d_i}} \left( \floor{0.5d_i} - m + 1 \right) P(v_i \in B_m) - n
\]

\[
= 2pn + 2 \sum_{i=1}^{n} \sum_{m=0}^{\floor{0.5d_i}} \left( \floor{0.5d_i} - m + 1 \right) \binom{d_i + 1}{m} p^m (1-p)^{d_i+1-m} - n
\]

\[
\leq 2pn + 2 \sum_{i=1}^{n} \max_{d_i \geq \delta} f(d_i, p) - n.
\]

It is not difficult to check that

\[
2(1-p)(2p + (1-p)(d + 4)/(d + 2)) < 1
\]

for any \(d \geq \delta \geq 2\). By Lemma 1,

\[
\max_{d \geq \delta} f(d, p) \in \{ f(\delta, p), f(\delta + 1, p) \}.
\]

The last inequality implies \(2(1-p) < 1\). Therefore, by Lemma 1,

\[
\max_{d \geq \delta} f(d, p) = f(\delta, p)
\]

if \(\delta\) is odd. If \(\delta\) is even, then we can prove that

\[
\max_{d \geq \delta} f(d, p) = f(\delta + 1, p).
\]

Thus,

\[
\max_{d \geq \delta} f(d, p) = f(\delta', p).
\]
Therefore,

\[ E[f(V(G))] \leq 2pn + 2n \sum_{m=0}^{\lceil 0.5\delta' \rceil} (\lceil 0.5\delta' \rceil - m + 1) \binom{\delta' + 1}{m} p^m (1 - p)^{\delta' + 1 - m} - n. \]

Since \( (\lceil 0.5\delta' \rceil - m + 1) \binom{\delta' + 1}{m} \leq \binom{\delta' + 1}{\lceil 0.5\delta' \rceil} \binom{\lceil 0.5\delta' \rceil}{m} \),

we obtain

\[ E[f(V(G))] \leq 2pn + 2n \sum_{m=0}^{\lceil 0.5\delta' \rceil} \binom{\delta' + 1}{\lceil 0.5\delta' \rceil} \binom{\lceil 0.5\delta' \rceil}{m} p^m (1 - p)^{\delta' + 1 - m} - n \]

\[ = 2pn + 2n \binom{\delta' + 1}{\lceil 0.5\delta' \rceil} (1 - p)^{\delta' - \lceil 0.5\delta' \rceil + 1} \times \]

\[ \times \sum_{m=0}^{\lceil 0.5\delta' \rceil} \binom{\lceil 0.5\delta' \rceil}{m} p^m (1 - p)^{\lceil 0.5\delta' \rceil - m} - n \]

\[ = 2pn + 2n \tilde{d}_{0.5} (1 - p)^{\delta' - \lceil 0.5\delta' \rceil + 1} - n. \]

Taking into account that \( \delta' - \lceil 0.5\delta' \rceil = \lceil 0.5\delta' \rceil = \lceil 0.5\delta \rceil = \hat{\delta} \), we have

\[ E[f(V(G))] \leq 2pn + 2n \tilde{d}_{0.5} (1 - p)^{\hat{\delta} + 1} - n \]

\[ \leq \left( 1 - \frac{2 \hat{\delta}}{(1 + \hat{\delta})^{1+\hat{\delta}} \tilde{d}_{0.5}^{1/\hat{\delta}}} \right) n, \]

as required. The proof of Theorem 51 is complete. \( \blacksquare \)

Our next result and its corollaries give a modest improvement of Theorem 49. More precisely, the upper bound of Theorem 52 is asymptotically 1.63 times
better than the bound of Theorem 49 and for \( \delta = 10^6 \) the improvement is 1.44 times.

**Theorem 52** If \( \delta(G) \geq 10^6 \), then

\[
\gamma_s(G) \leq \frac{\sqrt{6 \ln(\delta + 1)} + 1.21}{\sqrt{\delta + 1}} n.
\]

**Proof:** Denote \( \delta^+ = \delta + 1 \), \( N_v = N[v] \) and \( n_v = |N_v| \). Let \( A \) be a set formed by an independent choice of vertices of \( G \), where each vertex is selected with the probability

\[
p = 0.5 + \sqrt{1.5 \ln \delta^+ / \delta^+}.
\]

Let us construct two sets \( Q \) and \( U \) in the following way: for each vertex \( v \in V(G) \), if \( |N_v \cap A| \leq 0.5n_v \), then we put \( v \in U \) and add \( \lfloor 0.5n_v + 1 \rfloor \) vertices of \( N_v \) to \( Q \). Furthermore, we assign “+” to \( A \cup Q \), and “–” to all other vertices. The resulting function \( g : V(G) \to \{-1, 1\} \) is a signed domination function, and

\[
g(V(G)) = 2|A \cup Q| - n \leq 2|A| + 2|Q| - n.
\]

The expectation of \( g(V(G)) \) is

\[
E[g(V(G))] \leq 2E[|A|] + 2E[|Q|] - n
\]

\[
= 2pn - n + 2E[|Q|]. \quad (5.4)
\]

It is easy to see that \( |Q| \leq \sum_{v \in U} \lfloor 0.5n_v + 1 \rfloor \), hence

\[
E[|Q|] \leq \sum_{v \in V(G)} \lfloor 0.5n_v + 1 \rfloor P[v \in U], \quad (5.5)
\]
5.2 Upper Bounds for the Signed Domination Number

where

\[ P[v \in U] = P[|N_v \cap A| \leq 0.5n_v]. \]

Let us define the following random variables

\[ X_w = \begin{cases} 
1 - p & \text{if } w \in A \\
- p & \text{if } w \notin A 
\end{cases} \]

and denote \( X^* = \sum_{w \in N_v} X_w \). We have

\[ |N_v \cap A| \leq 0.5n_v \quad \text{if and only if} \quad X^* \leq (1 - p)0.5n_v + (-p)0.5n_v. \]

Thus,

\[ P[|N_v \cap A| \leq 0.5n_v] = P[X^* \leq (0.5 - p)n_v]. \]

By Lemma 2,

\[ P[X^* \leq (0.5 - p)n_v] < e^{-\frac{1.5n_v \ln \delta^+/\delta^+}{1 + \sqrt{6 \ln \delta^+/\delta^+}}}. \]

For \( n_v \geq \delta^+ > 10^6 \), let us define

\[ y(n_v, \delta^+) = \frac{1.5n_v \ln \delta^+/\delta^+}{1 + \sqrt{6 \ln \delta^+/\delta^+}} - \ln(2.25n_v^{1.5}) + 1. \]

The function \( y(n_v, \delta^+) \) is an increasing function of \( n_v \) and \( y(\delta^+, \delta^+) > 0 \) for \( \delta^+ > 10^6 \). Hence \( y(n_v, \delta^+) \geq y(\delta^+, \delta^+) > 0 \) and

\[ \frac{1.5n_v \ln \delta^+/\delta^+}{1 + \sqrt{6 \ln \delta^+/\delta^+}} > \ln(2.25n_v^{1.5}) - 1. \]
5.2 Upper Bounds for the Signed Domination Number

We obtain
\[ P[|N_v \cap A| \leq 0.5n_v] < e^{1-\ln(2.25n_v^{1.5})} = \frac{e}{2.25n_v^{1.5}}, \]
and, using inequality (5.5),
\[ 2E[|Q|] \leq 2 \sum_{v \in V(G)} \frac{e(0.5n_v + 1)}{2.25n_v^{1.5}} \leq \frac{e(\delta + 3)n}{2.25(\delta + 1)^{1.5}} \leq \frac{1.21}{\sqrt{\delta + 1}}n. \]

Thus, (5.4) yields
\[ E[g(V(G))] \leq 2pn - n + \frac{1.21n}{\sqrt{\delta + 1}} = \frac{\sqrt{6\ln(\delta + 1) + 1.21}}{\sqrt{\delta + 1}}n, \]
as required. The proof of Theorem 52 is complete.

\[ \text{Corollary 10} \quad \text{If } 24,000 \leq \delta, \text{ then} \]
\[ \gamma_s(G) \leq \frac{\sqrt{6.8\ln(\delta + 1) + 0.32}}{\sqrt{\delta + 1}}n. \]

**Proof:** Putting \( p = 0.5 + \sqrt{1.7\ln \delta^+ / \delta^+} \) in the proof of Theorem 52, we obtain by Lemma 2
\[ P[X_v^* \leq (0.5 - p)n_v] < e^{-\frac{1.7n_v\ln \delta^+ / \delta^+}{1 + \sqrt{6.8\ln \delta^+ / \delta^+}}}. \]

Let us define the following function:
\[ y(n_v, \delta^+) = \frac{1.7n_v \ln \delta^+ / \delta^+}{1 + \sqrt{6.8\ln \delta^+ / \delta^+}} - \ln(3.14n_v^{1.5}) \]
5.2 Upper Bounds for the Signed Domination Number

for $n_v \geq \delta^+ > 24,000$. The function $y(n_v, \delta^+)$ is an increasing function of $n_v$ and $y(\delta^+, \delta^+) > 0$ for $\delta^+ > 24,000$. Hence $y(n_v, \delta^+) \geq y(\delta^+, \delta^+) > 0$ and

$$\frac{1.7n_v \ln \delta^+/\delta^+}{1 + \sqrt{6.8 \ln \delta^+/\delta^+}} > \ln(3.14n_v^{1.5}).$$

We obtain

$$2E[|Q|] \leq 2 \sum_{v \in V(G)} \frac{0.5n_v + 1}{3.14n_v^{1.5}} \leq \frac{(\delta + 3)n}{3.14(\delta + 1)^{1.5}} \leq \frac{0.32}{\sqrt{\delta + 1}} n.$$

Thus, (5.4) yields

$$E[g(V(G))] \leq 2pn - n + \frac{0.32n}{\sqrt{\delta + 1}} = \frac{\sqrt{6.8 \ln(\delta + 1) + 0.32}}{\sqrt{\delta + 1}} n,$$

as required. The proof is complete.

Corollary 11 If $1,000 \leq \delta \leq 24,000$, then

$$\gamma_s(G) \leq \sqrt{\ln(\delta + 1)(11.8 - 0.48 \ln \delta)} + 0.25 n.$$

Proof: It is similar to the proof of Corollary 10 if we put

$$p = 0.5 + \sqrt{(2.95 - 0.12 \ln \delta) \ln \delta^+/\delta^+}.$$
and consider the following function for $1,000 \leq \delta \leq 24,000$:

$$y(n_v, \delta^+) = \frac{(2.95 - 0.12 \ln \delta)n_v \ln \delta^+ / \delta^+}{1 + \sqrt{(11.8 - 0.48 \ln \delta) \ln \delta^+ / \delta^+}} - \ln(4.01n_v^{1.5}).$$

\[ \blacksquare \]

**Corollary 12** If $230 \leq \delta \leq 1,000$, then

$$\gamma_s(G) \leq \frac{\sqrt{\ln(\delta + 1)(18.16 - 1.4 \ln \delta)}}{\sqrt{\delta + 1}} + 0.25n.$$

**Proof:** It is similar to the proof of Corollary 10 if we put

$$p = 0.5 + \sqrt{(4.54 - 0.35 \ln \delta) \ln \delta^+ / \delta^+}$$

and consider the following function for $230 \leq \delta \leq 1,000$:

$$y(n_v, \delta^+) = \frac{(4.54 - 0.35 \ln \delta)n_v \ln \delta^+ / \delta^+}{1 + \sqrt{(18.16 - 1.4 \ln \delta) \ln \delta^+ / \delta^+}} - \ln(4.04n_v^{1.5}).$$

\[ \blacksquare \]

We believe that Füredi–Mubayi’s conjecture, saying that $\gamma_s(G) \leq \frac{Cn}{\sqrt{\delta}}$, is true for some small constant $C$. However, as the Peterson graph shows, $C > 1$, i.e. the behaviour of this bound is not good for relatively small values of $\delta$. Therefore, we propose the following rectified conjecture, which, roughly speaking, consists of two functions for ‘small’ and ‘large’ values of $\delta$. 

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5.2 Upper Bounds for the Signed Domination Number

**Conjecture 2** For some $C \leq 10$ and $\alpha$, $0.18 \leq \alpha < 0.21$,

$$\gamma_s(G) \leq \min \left\{ \frac{n}{\delta^\alpha}, \frac{Cn}{\sqrt{\delta}} \right\}. \quad (5.6)$$

The above results imply that if $C = 10$ and $\alpha = 0.13$, then this upper bound is true for all graphs with $\delta \leq 96 \times 10^4$.

5.2.1 Comparison of the Results

In this subsection, we provide figures, which draw a comparison among the upper bounds of Füredi–Mubayi’s result (see Theorem 49), Füredi–Mubayi’s conjecture (5.2), the new conjecture (5.6) and our results (combination of Theorem 51 and Corollary 12). In these figures, we consider $\delta$ in the range $2 \leq \delta \leq 520$.

![Figure 5.1: Comparison of the results on signed domination](image)

Figure 5.1 shows that our results provide a better upper bound than Theorem 49 for $\delta \geq 99$. For example, if $\delta = 99$, then by Theorem 49, $\gamma_s \leq 0.869n$, and
by Theorem 5.1 \( \gamma_s \leq 0.537n \). Note that Füredi and Mubayi do not provide any upper bounds for \( 30 \leq \delta < 99 \), while our result does. However, for very small values of \( \delta (\delta < 30) \), they found the best upper bounds.

In Figure 5.2, we can see line graphs illustrating the upper bound of Füredi–Mubayi’s conjecture (5.2) when \( C = 6 \) and \( C = 1 \). If \( C = 1 \), then the upper bound of this conjecture is good. However, the Peterson graph shows that \( C > 1 \), so the behaviour of this bound is not good for relatively small values of \( \delta \). This is illustrated in Figure 5.2 for \( C = 6 \), where for small values of \( \delta \) the upper bound exceeds \( n \).

Figure 5.3 shows the line graph of the new proposed conjecture when \( \alpha = 0.18 \). In this figure, we clearly see that the new conjecture (5.6) rectifies Füredi–Mubayi’s conjecture (5.2) for relatively small values of \( \delta \). The new conjecture provides better bounds when \( \delta \) is not very small.
5.3 A Lower Bound for the Signed Domination Number

The following theorem provides a lower bound for the signed domination number of a graph $G$ depending on its order and a parameter $\lambda$, which is determined on the basis of the degree sequence of $G$ (note that $\lambda$ may be equal to 0, in this case we put $\sum_{i=1}^{\lambda} = 0$). This result improves the bound of Theorem 50 and, in some cases, it provides a much better lower bound. For example, let us consider a graph $G$ consisting of two vertices of degree 5 and $n - 2$ vertices of degree 3. Then, by Theorem 50

$$\gamma_s(G) \geq 0.25n - 1,$$

while Theorem 53 yields

$$\gamma_s(G) \geq 0.5n - 1.$$
5.3 A Lower Bound for the Signed Domination Number

**Theorem 53** Let $G$ be a graph with $n$ vertices and degrees $d_1 \leq d_2 \leq \ldots \leq d_n$. Then

$$\gamma_s(G) \geq n - 2\lambda,$$

where $\lambda \geq 0$ is the largest integer such that

$$\sum_{i=1}^{\lambda} \left\lfloor \frac{d_i}{2} + 1 \right\rfloor \leq \sum_{i=\lambda+1}^{n} \left\lfloor \frac{d_i}{2} \right\rfloor.$$

**Proof:** Let $f$ be a signed domination function of minimum weight of the graph $G$. Let us denote

$$X = \{v \in V(G) : f(v) = -1\},$$

and

$$Y = \{v \in V(G) : f(v) = 1\}.$$

We have

$$\gamma_s(G) = f(V(G)) = |Y| - |X| = n - 2|X|.$$

By definition, for any vertex $v$,

$$f[v] = \sum_{u \in N[v]} f(u) \geq 1.$$

Therefore, for all $v \in V(G)$,

$$|N[v] \cap Y| - |N[v] \cap X| \geq 1.$$
5.3 A Lower Bound for the Signed Domination Number

Using this inequality, we obtain

\[ |N[v]| = \deg(v) + 1 = |N[v] \cap Y| + |N[v] \cap X| \leq 2|N[v] \cap Y| - 1. \]

Hence

\[ |N[v] \cap Y| \geq \frac{\deg(v)}{2} + 1. \]

Since \(|N[v] \cap Y|\) is an integer, we conclude

\[ |N[v] \cap Y| \geq \left\lceil \frac{\deg(v)}{2} \right\rceil + 1 \]

and

\[ |N[v] \cap X| = \deg(v) + 1 - |N[v] \cap Y| \leq \left\lfloor \frac{\deg(v)}{2} \right\rfloor. \]

Denote by \(e_{X,Y}\) the number of edges between the parts \(X\) and \(Y\). We have

\[ e_{X,Y} = \sum_{v \in X} |N[v] \cap Y| \geq \sum_{v \in X} \left( \left\lceil \frac{\deg(v)}{2} \right\rceil + 1 \right) \geq \sum_{i=1}^{\frac{|X|}{2}} \left( \left\lceil \frac{d_i}{2} \right\rceil + 1 \right). \]

Note that if \(X = \emptyset\), then we put \(\sum_{i=1}^0 g(i) = 0\). On the other hand,

\[ e_{X,Y} = \sum_{v \in Y} |N[v] \cap X| \leq \sum_{v \in Y} \left\lceil \frac{\deg(v)}{2} \right\rceil \leq \sum_{i=n-|Y|+1}^{n} \left( \left\lfloor \frac{d_i}{2} \right\rfloor \right) = \sum_{i=|X|+1}^{n} \left( \left\lfloor \frac{d_i}{2} \right\rfloor \right).

Therefore, the following inequality holds:

\[ \sum_{i=1}^{\frac{|X|}{2}} \left( \left\lceil \frac{d_i}{2} \right\rceil + 1 \right) \leq \sum_{i=|X|+1}^{n} \left\lfloor \frac{d_i}{2} \right\rfloor. \]
5.3 A Lower Bound for the Signed Domination Number

Since $\lambda \geq 0$ is the largest integer such that

$$\sum_{i=1}^{\lambda} \left(\left\lfloor \frac{d_i}{2} \right\rfloor + 1 \right) \leq \sum_{i=\lambda+1}^{n} \left\lfloor \frac{d_i}{2} \right\rfloor,$$

we conclude that

$$|X| \leq \lambda.$$

Thus,

$$\gamma_s(G) = n - 2|X| \geq n - 2\lambda.$$

The proof is complete.

Theorem 53 immediately implies the following known results:

**Corollary 13 ([41] and [86])** For any graph $G$,

$$\gamma_s(G) \geq \left(\left\lceil 0.5\delta \right\rceil - \left\lfloor 0.5\Delta \right\rfloor + 1 \right) n.$$

Note that Haas and Wexler [41] formulated the above bound only for graphs with $\delta \geq 2$, while Zhang et al. [86] proved a weaker version without the ceiling and floor functions.

**Corollary 14 ([51])** If $\delta$ is odd and $G$ is $\delta$-regular, then

$$\gamma_s(G) \geq \frac{2n}{\delta + 1}.$$

**Corollary 15 ([29])** If $\delta$ is even and $G$ is $\delta$-regular, then

$$\gamma_s(G) \geq \frac{n}{\delta + 1}.$$
Disjoint unions of complete graphs show that these lower bounds are sharp whenever $n/(\delta + 1)$ is an integer, and therefore the bound of Theorem 53 is sharp for regular graphs.
Chapter 6

Conclusions

The aim of this thesis is to provide new upper bounds for multiple and other domination parameters by means of the probabilistic method. The estimation method of first moments described in detail in [33] is used in this research. The main focus of this thesis lies on the study of dominating sets in graphs, in particular multiple domination parameters. The following domination parameters have been studied in this thesis: $k$-domination, $k$-tuple domination, $k$-total domination, $\alpha$-domination, $\alpha$-rate domination, Roman domination, global domination, restrained domination and signed domination. New upper bounds for these parameters have been derived and comparisons with other well-known results have been drawn.

We conclude by giving an overview of the new results provided in this work, posing open problems and, finally, by discussing horizons for future research.

6.1 Brief Summary of Results

This section contains a brief listing of the main new results presented in this thesis. For a detailed summary of the existing well-known results we refer to
6.1 Brief Summary of Results

Subsection 1.2.3.

Results for $k$-Tuple Domination and $k$-Domination

In Theorems 30 and 31, the following upper bounds are proved:

For any graph $G$ with $\delta \geq k$,

$$\gamma_{\times k}(G) \leq \left(1 - \frac{\delta'}{(d_{k-1} + d_{k-2})^{1/\delta'}(1 + \delta')^{1+1/\delta'}}\right)n,$$

where $\delta' = \delta - k + 1$.

For any graph $G$ with $\delta \geq k$,

$$\gamma_k(G) \leq \left(1 - \frac{\delta'}{d_{k-1}^{1/\delta'}(1 + \delta')^{1+1/\delta'}}\right)n,$$

where $\delta' = \delta - k + 1$.

Results for $\alpha$-Domination and $\alpha$-Rate Domination

We denote

$$\hat{d}_\alpha = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{d_i}{\lceil \alpha d_i \rceil} - 1\right)$$

and

$$\tilde{d}_\alpha = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{d_i + 1}{\lceil \alpha d_i \rceil} - 1\right),$$

see Chapter 3 for details. In Theorems 32 and 33, the following upper bounds are proved:
6.1 Brief Summary of Results

For any graph $G$,

$$
\gamma_\alpha(G) \leq \left( 1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1+1/\delta} \hat{d}_{\alpha}^{1/\delta}} \right) n,
$$

where $\hat{\delta} = \lceil \delta(1 - \alpha) \rceil + 1$.

For any graph $G$ and $0 < \alpha \leq 1$,

$$
\gamma_{\times \alpha}(G) \leq \left( 1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1+1/\delta} \tilde{d}_{\alpha}^{1/\delta}} \right) n,
$$

where $\tilde{\delta} = \lceil \delta(1 - \alpha) \rceil + 1$.

Results for Global and Roman Domination

In Theorems 42 and 44, the following new upper bounds for the global and Roman domination numbers are derived:

For any graph $G$ with $\delta' = \min\{\delta, \bar{\delta}\} > 0$,

$$
\gamma_g(G) \leq \left( 1 - \frac{\delta'}{2^{1/\delta'} (1 + \delta')^{1+1/\delta'}} \right) n.
$$

For any graph $G$ with $\delta > 0$,

$$
\gamma_R(G) \leq 2 \left( 1 - \frac{2^{1/\delta} \delta}{(1 + \delta)^{1+1/\delta}} \right) n.
$$

Then, by using a modification of Alon’s construction, we also prove that the results above are asymptotically best possible (see Theorems 43 and 45, respectively).
6.1 Brief Summary of Results

When \( n \) is large, there exists a graph \( G \) such that

\[
\gamma_g(G) \geq \frac{\ln(\delta' + 1) + \ln 2 + 1}{\delta' + 1}n(1 + o(1)).
\]

When \( n \) is large, there exists a graph \( G \) such that

\[
\gamma_R(G) \geq \frac{2 \ln(\delta + 1) - \ln 4 + 2}{\delta + 1}n(1 + o(1)).
\]

Results for Restrained and Total Restrained Domination

In Theorems 46 and 47 we prove the following upper bounds for the restrained and total restrained domination numbers:

For almost every graph, \( \gamma_r(G) = \gamma(G) \) and \( \gamma_{tr}(G) = \gamma_t(G) \). Moreover,

\[
\gamma_r(G) = \lfloor \log n - 2 \log \log n + \log \log e \rfloor + \epsilon,
\]

where \( \epsilon = 1 \) or \( 2 \), and \( \log \) denotes the logarithm with base \( 1/q \).

For any graph \( G \) with \( \delta > 0 \),

\[
\gamma_r(G) \leq \frac{2 \ln(\delta + 1) + \delta + 3}{\delta + 1}n - 2\beta_1
\]

and

\[
\gamma_{tr}(G) \leq \frac{2 \ln \delta + \delta + 2}{\delta}n - 2\beta_1,
\]

where \( \beta_1 \) is the matching number of \( G \).
Results for Signed Domination

We denote
\[ \tilde{d}_{0.5} = \left( \delta' + 1 \right) \left[ 0.5 \delta' \right], \]
where
\[ \delta' = \begin{cases} \delta & \text{if } \delta \text{ is odd;} \\
\delta + 1 & \text{if } \delta \text{ is even}. \end{cases} \]

In Theorems 51 and 52, the following upper bounds are derived:

For any graph \( G \) with \( \delta > 1 \),
\[
\gamma_s(G) \leq \left(1 - \frac{2\hat{\delta}}{(1 + \hat{\delta})^{1+1/\hat{\delta}} \tilde{d}_{0.5}}\right)n,
\]
where \( \hat{\delta} = \left\lfloor 0.5\delta \right\rfloor \).

If \( \delta(G) \geq 10^6 \), then
\[
\gamma_s(G) \leq \frac{\sqrt{6 \ln(\delta + 1)} + 1.21}{\sqrt{\delta + 1}}n.
\]

Additionally, we proposed the following rectified conjecture, which, roughly speaking, consists of two functions for ‘small’ and ‘large’ values of \( \delta \):

For some \( C \leq 10 \) and \( 0.18 \leq \alpha < 0.21 \),
\[
\gamma_s(G) \leq \min \left\{ \frac{n}{\delta^\alpha}, \frac{Cn}{\sqrt{\delta}} \right\}.
\]

Also, for the signed domination number, a new lower bound is given in Theorem 53.
Let $G$ be a graph with $n$ vertices and degrees $d_1 \leq d_2 \leq \ldots \leq d_n$. Then

$$\gamma_s(G) \geq n - 2\lambda,$$

where $\lambda \geq 0$ is the largest integer such that

$$\sum_{i=1}^{\lambda} \left\lceil \frac{d_i}{2} + 1 \right\rceil \leq \sum_{i=\lambda+1}^{n} \left\lfloor \frac{d_i}{2} \right\rfloor.$$

### 6.2 Open Problems and Future Research

This thesis provides different starting points for future research. We formulate some of the prospects in this area.

There are many other domination parameters, which are out of scope of this thesis. It would be interesting to use a probabilistic construction to obtain upper bounds for those other domination parameters. For example, upper bounds for $\alpha$-independent domination number can be derived.

In this thesis, we proved that new upper bounds for the global and Roman domination parameters presented here are asymptotically best possible. We wonder if the same can be proved for the upper bounds of other domination parameters, which were derived by using the same approach.

We also wonder if it is possible to derandomize algorithms presented here for $\alpha$-domination, $\alpha$-rate domination, $k$-domination and $k$-tuple domination parameters or to obtain independent deterministic algorithms to find the corresponding dominating sets satisfying the upper bounds of our results. The designing of effective randomized algorithms for finding corresponding dominating sets, whose
expected sizes satisfy the upper bounds of other dominating parameters presented in this thesis, can be considered as a topic for future research. Algorithms approximating the above mentioned domination parameters to a certain degree of precision would be interesting as well. For the $k$-tuple domination number, an interesting approximation algorithm was found by Klasing and Laforest [56].

During our research we have formulated open problems for the global, Roman, restrained and total restrained domination numbers, see Problems 1, 2 and 3 in Section 4.5, which can be considered a subject for future research.

It would be also interesting to prove Conjecture 2 which was proposed in Chapter 5 for the signed domination number. This conjecture is a rectified version of Füredi–Mubayi’s conjecture [37].
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Appendix A

Upper bounds for the $\alpha$-domination parameters

Upper Bounds for $\alpha$-Domination Parameters

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Abstract

In this paper, we provide a new upper bound for the $\alpha$-domination number. This result generalises the well-known Caro-Roditty bound for the domination number of a graph. The same probabilistic construction is used to generalise another well-known upper bound for the classical domination in graphs. We also prove similar upper bounds for the $\alpha$-rate domination number, which combines the concepts of $\alpha$-domination and $k$-tuple domination.

Keywords: Graph; Domination; $\alpha$-Domination; $\alpha$-Rate Domination

1 Introduction

Domination is one of the fundamental concepts in graph theory with various applications to ad hoc networks, biological networks, distributed computing, social networks and web graphs [1, 6, 8, 13]. Dominating sets in graphs are natural models for facility location problems in operational research. An important role is played by multiple domination, for example $k$-dominating sets can be used for balancing efficiency and fault tolerance [8].

We consider undirected simple finite graphs. If $G$ is a graph of order $n$, then $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices of $G$ and $d_i$ denotes the degree of $v_i$. Let $N(v)$ denote the neighbourhood of a vertex $v$ in $G$, and $N[v] = N(v) \cup \{v\}$ be the closed neighbourhood of $v$. A set $X \subseteq V(G)$ is called a dominating set if every vertex not in $X$ is adjacent to at least one vertex in $X$. The minimum cardinality of a dominating set of $G$ is the domination number $\gamma(G)$. A set $X$ is called a $k$-dominating set if every vertex not in $X$ has at least $k$ neighbors in $X$. The minimum cardinality of a $k$-dominating set of $G$ is the $k$-domination number $\gamma_k(G)$. A set $X$ is called a $k$-tuple dominating set of $G$ if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. The minimum cardinality of a $k$-tuple dominating set of $G$ is the $k$-tuple domination number $\gamma_{\times k}(G)$. The $k$-tuple domination number is only defined for graphs with $\delta \geq k - 1$. A number of upper bounds for the multiple domination numbers can be found in [5, 10, 11, 12, 17].
Let $\alpha$ be a real number satisfying $0 < \alpha \leq 1$. A set $X \subseteq V(G)$ is called an $\alpha$-dominating set of $G$ if for every vertex $v \in V(G) - X$, $|N(v) \cap X| \geq \alpha d_v$, i.e. $v$ is adjacent to at least $\lceil \alpha d_v \rceil$ vertices of $X$. The minimum cardinality of an $\alpha$-dominating set of $G$ is called the $\alpha$-domination number $\gamma_\alpha(G)$. The $\alpha$-domination was introduced by Dunbar et al. [9]. It is easy to see that $\gamma(G) \leq \gamma_\alpha(G)$, and $\gamma_\alpha_1(G) \leq \gamma_\alpha_2(G)$ for $\alpha_1 < \alpha_2$. Also, $\gamma(G) = \gamma_\alpha(G)$ if $\alpha$ is sufficiently close to 0.

For an arbitrary graph $G$ with $n$ vertices and $m$ edges, denote by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ the minimum and maximum vertex degrees of $G$, respectively. The following results are proved in [9]:

$$\frac{\alpha \delta n}{\Delta + \alpha \delta} \leq \gamma_\alpha(G) \leq \frac{\Delta n}{\Delta + (1 - \alpha) \delta} \quad (1)$$

and

$$\frac{2 \alpha m}{(1 + \alpha) \Delta} \leq \gamma_\alpha(G) \leq \frac{(2 - \alpha) \Delta n - (2 - 2 \alpha) m}{(2 - \alpha) \Delta} \quad (2)$$

Interesting results on $\alpha$-domination perfect graphs can be found in [7]. The problem of deciding whether $\gamma_\alpha(G) \leq k$ for a positive integer $k$ is known to be NP-complete [9]. Therefore, it is important to have good upper bounds for the $\alpha$-domination number and efficient approximation algorithms for finding ‘small’ $\alpha$-dominating sets.

For $0 < \alpha \leq 1$, the $\alpha$-degree of a graph $G$ is defined as follows:

$$\hat{d}_\alpha = \hat{d}_\alpha(G) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d_i}{\lceil \alpha d_i \rceil} - 1 \right).$$

In this paper, we use a probabilistic approach to prove that

$$\gamma_\alpha(G) \leq \left( 1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1 + 1/\hat{\delta}} \hat{d}_\alpha^{1/\hat{\delta}}} \right) n,$$

where $\hat{\delta} = \lceil \delta(1 - \alpha) \rceil + 1$. This result generalises the well-known upper bound of Caro and Roditty ([13], p. 48). Using the same probabilistic construction, we also show that

$$\gamma_\alpha(G) \leq \frac{\ln(\hat{\delta} + 1) + \ln \hat{d}_\alpha + 1}{\hat{\delta} + 1} n,$$

which generalises another well-known upper bound of Alon and Spencer [3], Arnautov [4], Lovász [15] and Payan [16]. Finally, we introduce the $\alpha$-rate domination number, which combines together the concepts of $\alpha$-domination and $k$-tuple domination, and show that the $\alpha$-rate domination number satisfies two similar upper bounds. The random constructions used in this paper also provide probabilistic algorithms to find $\alpha$-dominating and $\alpha$-rate dominating sets satisfying corresponding bounds.

### 2 New Upper Bounds for the $\alpha$-Domination Number

One of the strongest known upper bounds for the domination number is due to Caro and Roditty:

**Theorem 1 (Caro and Roditty [13], p. 48)** For any graph $G$ with $\delta \geq 1$,

$$\gamma(G) \leq \left( 1 - \frac{\delta}{(1 + \delta)^{1 + 1/\delta}} \right) n. \quad (3)$$
The upper bound (3) is generalised for the $\alpha$-domination number in Theorem 2. Indeed, if $d_i$ are fixed for all $i = 1, \ldots, n$, and $\alpha$ is sufficiently close to 0, then $\hat{\delta} = \delta$ (provided $\delta \geq 1$) and $\hat{d}_\alpha = 1$.

**Theorem 2** For any graph $G$,

$$
\gamma_\alpha(G) \leq \left(1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1/\hat{d}_\alpha}}\right)n,
$$

(4)

where $\hat{\delta} = \lfloor\delta(1-\alpha)\rfloor + 1$.

**Proof:** Let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with the probability

$$
p = 1 - \left(\frac{1}{(1 + \hat{\delta})\hat{d}_\alpha}\right)^{1/\hat{\delta}}.
$$

(5)

Let us denote

$$
B = \{v_i \in V(G) - A : |N(v_i) \cap A| \leq \lceil\alpha d_i\rceil - 1\}.
$$

It is obvious that the set $D = A \cup B$ is an $\alpha$-dominating set. The expectation of $|D|$ is

$$
E(|D|) = E(|A|) + E(|B|)
$$

$$
= \sum_{i=1}^{n} P(v_i \in A) + \sum_{i=1}^{n} P(v_i \in B)
$$

$$
= pn + \sum_{i=1}^{n} \sum_{r=0}^{\lceil\alpha d_i\rceil-1} \binom{d_i}{r} p^r (1-p)^{d_i-r+1}.
$$

It is easy to see that, for $0 \leq r \leq \lceil\alpha d_i\rceil - 1$,

$$
\binom{d_i}{r} \leq \binom{d_i}{\lceil\alpha d_i\rceil - 1} \binom{\lceil\alpha d_i\rceil - 1}{r}.
$$

Also,

$$
d_i - \lceil\alpha d_i\rceil \geq \lfloor\delta(1-\alpha)\rfloor.
$$

Therefore,

$$
E(|D|) \leq pn + \sum_{i=1}^{n} \binom{d_i}{\lceil\alpha d_i\rceil - 1} (1-p)^{d_i-\lceil\alpha d_i\rceil+2} \times
$$

$$
\sum_{r=0}^{\lceil\alpha d_i\rceil-1} \binom{\lceil\alpha d_i\rceil - 1}{r} p^r (1-p)^{\lceil\alpha d_i\rceil-1-r}
$$

$$
= pn + \sum_{i=1}^{n} \binom{d_i}{\lceil\alpha d_i\rceil - 1} (1-p)^{d_i-\lceil\alpha d_i\rceil+2}
$$

$$
\leq pn + (1-p)^{\lfloor\delta(1-\alpha)\rfloor+2}\hat{d}_\alpha n
$$

$$
= pn + (1-p)^{\hat{\delta}+1}\hat{d}_\alpha n
$$

$$
= \left(1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1/\hat{d}_\alpha}}\right)n.
$$

(6)
Note that the value of $p$ in (5) is chosen to minimize the expression (6). Since the expectation is an average value, there exists a particular $\alpha$-dominating set of order at most 

$$\left(1 - \frac{\tilde{d}}{(1+\delta)^{1/\delta}}\right) n,$$

as required. The proof of the theorem is complete.

Notice that in some cases Theorem 2 provides a much better bound than the upper bound in (1). For example, if $G$ is a 1000-regular graph, then Theorem 2 gives $\gamma_{0.1}(G) < 0.305n$, while (1) yields only $\gamma_{0.1}(G) < 0.527n$.

**Corollary 1** For any graph $G$,

$$\gamma_\alpha(G) \leq \frac{\ln(\delta + 1) + \ln \tilde{d}_\alpha + 1}{\delta + 1} n. \quad (7)$$

**Proof:** We put

$$p = \min \left\{ 1, \frac{\ln(\delta + 1) + \ln \tilde{d}_\alpha}{\delta + 1} \right\}.$$

Using the inequality $1 - p \leq e^{-p}$, we can estimate the expression (6) as follows:

$$E(|D|) \leq pn + e^{-p(\delta+1)} \tilde{d}_\alpha n.$$

If $p = 1$, then the result easily follows. If $p = \frac{\ln(\delta+1)+\ln \tilde{d}_\alpha}{\delta+1}$, then

$$E(|D|) \leq \frac{\ln(\delta + 1) + \ln \tilde{d}_\alpha + 1}{\delta + 1} n,$$

as required.

Corollary 1 generalises the following well-known upper bound independently proved by several authors [3, 4, 15, 16]:

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n. \quad (8)$$

### 3 $\alpha$-Rate Domination

Define a set $X \subseteq V(G)$ to be an $\alpha$-rate dominating set of $G$ if for any vertex $v \in V(G)$,

$$|N[v] \cap X| \geq \alpha d_v.$$

Let us call the minimum cardinality of an $\alpha$-rate dominating set of $G$ the $\alpha$-rate domination number $\gamma_{\times \alpha}(G)$. It is easy to see that $\gamma_\alpha(G) \leq \gamma_{\times \alpha}(G)$. The concept of $\alpha$-rate domination is similar to the well-known $k$-tuple domination (for example, see [14, 17]). For $0 < \alpha \leq 1$, the closed $\alpha$-degree of a graph $G$ is defined as follows:

$$\tilde{d}_\alpha = \tilde{d}_\alpha(G) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d_i + 1}{\lceil \alpha d_i \rceil - 1} \right).$$

In fact, the only difference between the $\alpha$-degree and the closed $\alpha$-degree is that to compute the latter we choose from $d_i + 1$ vertices instead of $d_i$, i.e. from the closed neighborhood $N[v_i]$ of $v_i$ instead of $N(v_i)$.

The following theorem provides an analogue of the Caro-Roditty bound (Theorem 1) for the $\alpha$-rate domination number:
Theorem 3 For any graph $G$ and $0 < \alpha \leq 1$,

$$\gamma_{\times \alpha}(G) \leq \left(1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1+1/\hat{\delta}}} \frac{\tilde{d}^{1/\hat{\delta}}}{\alpha}\right) n,$$  

(9)

where $\hat{\delta} = \lceil \delta(1 - \alpha) \rceil + 1$.

Proof: Let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with probability $p$, $0 \leq p \leq 1$. For $m \geq 0$, denote by $B_m$ the set of vertices $v \in V(G)$ dominated by exactly $m$ vertices of $A$ and such that $|N[v] \cap A| < \alpha d_v$, i.e.

$$|N[v] \cap A| = m \leq \lceil \alpha d_v \rceil - 1.$$

Note that each vertex $v \in V(G)$ is in at most one of the sets $B_m$ and $0 \leq m \leq \lceil \alpha d_v \rceil - 1$. We form a set $B$ in the following way: for each vertex $v \in B_m$, select $\lceil \alpha d_v \rceil - m$ vertices from $N(v)$ that are not in $A$ and add them to $B$. Consider the set $D = A \cup B$. It is easy to see that $D$ is an $\alpha$-rate dominating set. The expectation of $|D|$ is:

$$\mathbf{E}(|D|) \leq \mathbf{E}(|A|) + \mathbf{E}(|B|)$$

$$\leq \sum_{i=1}^{n} P(v_i \in A) + \sum_{i=1}^{n} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} \left(\begin{array}{c} \lceil \alpha d_i \rceil - 1 \\ m \end{array}\right) p^m (1 - p)^{d_i + 1 - m}$$

$$= pn + \sum_{i=1}^{n} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} \left(\begin{array}{c} \lceil \alpha d_i \rceil - 1 \\ m \end{array}\right) p^m (1 - p)^{d_i + 1 - m}$$

$$\leq pn + \sum_{i=1}^{n} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} \left(\begin{array}{c} \lceil \alpha d_i \rceil - 1 \\ m \end{array}\right) p^m (1 - p)^{\lceil \alpha d_i \rceil - 1 - m}$$

$$= pn + \sum_{i=1}^{n} \left(\begin{array}{c} \lceil \alpha d_i \rceil - 1 \\ \lceil \alpha d_i \rceil - 1 \end{array}\right) (1 - p)^{\lceil \alpha d_i \rceil + 1 - m}$$

since

$$\left(\begin{array}{c} \lceil \alpha d_i \rceil - m \\ m \end{array}\right) \leq \left(\begin{array}{c} \lceil \alpha d_i \rceil - 1 \\ \lceil \alpha d_i \rceil - 1 \end{array}\right) \left(\begin{array}{c} \lceil \alpha d_i \rceil - 1 \\ m \end{array}\right).$$

Thus,

$$\mathbf{E}(|D|) \leq pn + (1 - p)^{\hat{\delta} + 1} \tilde{d}_an.$$

Minimizing the expression (10) with respect to $p$, we obtain

$$\mathbf{E}(|D|) \leq \left(1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1+1/\hat{\delta}}} \frac{\tilde{d}^{1/\hat{\delta}}}{\alpha}\right) n.$$  

(10)

A6
as required. The proof of Theorem 3 is complete.

\[ \gamma_{\times\alpha}(G) \leq \frac{\ln(\hat{\delta} + 1) + \ln \tilde{d}_\alpha + 1}{\hat{\delta} + 1} n. \]  

(11)

\textbf{Proof:} Using an approach similar to that in the proof of Corollary 1, the result follows if we put

\[ p = \min \left\{ 1, \frac{\ln(\hat{\delta} + 1) + \ln \tilde{d}_\alpha}{\hat{\delta} + 1} \right\} \]

and use the inequality \( 1 - p \leq e^{-p} \) to estimate the expression (10) as follows:

\[ E(|D|) \leq pn + e^{-p(\hat{\delta}+1)}\tilde{d}_\alpha n. \]

Note that, similar to Corollary 1, the bound of Corollary 2 also generalises the classical upper bound (8). However, the probabilistic construction used to obtain the bounds (9) and (11) is different from that to obtain the bounds (4) and (7).

4 Final Remarks and Open Problems

Notice that the concept of the \( \alpha \)-rate domination number \( \gamma_{\times\alpha}(G) \) is ‘opposite’ to the \( \alpha \)-independent \( \alpha \)-domination number \( i_{\alpha}(G) \) as defined in [7]. It would be interesting to use a probabilistic method construction to obtain an upper bound for \( i_{\alpha}(G) \).

Also, the random constructions used to obtain the upper bounds (4), (7), (9) and (11) provide probabilistic algorithms to find corresponding dominating sets in a given graph \( G \). It would be interesting to derandomize these algorithms or to obtain independent deterministic algorithms to find corresponding dominating sets satisfying the upper bounds (4), (7), (9) and (11). Algorithms approximating the \( \alpha \) - and \( \alpha \)-rate domination numbers up to a certain degree of precision would be interesting too. For the \( k \)-tuple domination number, an interesting approximation algorithm was found by Klasing and Laforest [14].

Using probabilistic methods, Alon [2] proved that the bound (8) is asymptotically best possible. More precisely, it was proved that when \( n \) is large there exists a graph \( G \) such that

\[ \gamma(G) \geq \frac{\ln(\delta + 1) + 1}{\delta + 1} n(1 + o(1)). \]

We wonder if a similar result can be proved for the bounds (7) and (11), and conjecture that when \( n \) is large enough there exist graphs \( G \) and \( H \) such that

\[ \gamma_{\alpha}(G) \geq \frac{\ln(\hat{\delta} + 1) + \ln \tilde{d}_\alpha + 1}{\hat{\delta} + 1} n(1 + o(1)) \]

and

\[ \gamma_{\times\alpha}(H) \geq \frac{\ln(\hat{\delta} + 1) + \ln \tilde{d}_\alpha + 1}{\hat{\delta} + 1} n(1 + o(1)). \]
References


Appendix B

Discrepancy and Signed Domination in Graphs and Hypergraphs


Contributed talk based on this paper given at 22nd British Combinatorial Conference, July 5–10, 2009, University of St. Andrews, Scotland, UK.
Discrepancy and Signed Domination in Graphs
and Hypergraphs

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Abstract

For a graph $G$, a signed domination function of $G$ is a two-colouring of the vertices of $G$ with colours $+1$ and $-1$ such that the closed neighbourhood of every vertex contains more $+1$'s than $-1$'s. This concept is closely related to combinatorial discrepancy theory as shown by Füredi and Mubayi [J. Combin. Theory, Ser. B 76 (1999) 223–239]. The signed domination number of $G$ is the minimum of the sum of colours for all vertices, taken over all signed domination functions of $G$. In this paper, we present new upper and lower bounds for the signed domination number. These new bounds improve a number of known results.

Keywords: graphs, signed domination function, signed domination number.

1 Discrepancy Theory and Signed Domination

Originated from number theory, discrepancy theory is, generally speaking, the study of irregularities of distributions in various settings. Classical combinatorial discrepancy theory is devoted to the problem of partitioning the vertex set of a hypergraph into two classes in such a way that all hyperedges are split into approximately equal parts by the classes, i.e. we are interested in measuring the deviation of an optimal partition from perfect, when all hyperedges are split into equal parts. It may be pointed out that many classical results in various areas of mathematics, e.g. geometry and number theory, can be formulated in these terms. Combinatorial discrepancy theory was introduced and studied by Beck in [3]. Also, studies on discrepancy theory have been conducted in [2, 4, 5] and [17].

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with the vertex set $V$ and the hyperedge set $\mathcal{E} = \{E_1, ..., E_m\}$. One of the main problems in classical combinatorial discrepancy theory is to colour the elements of $V$ by two colours in such a way that all of the hyperedges have almost the same number of elements of each colour. Such a partition of $V$ into two classes can be represented by a function

$$f : V \rightarrow \{+1, -1\}.$$

For a set $E \subseteq V$, let us define the imbalance of $E$ as follows:

$$f(E) = \sum_{v \in E} f(v).$$

B2
First defined by Beck [3], the discrepancy of $H$ with respect to $f$ is

$$D(H, f) = \max_{E_i \in \mathcal{E}} |f(E_i)|$$

and the discrepancy of $H$ is

$$D(H) = \min_{f : V \to \{-1, +1\}} D(H, f).$$

Thus, the discrepancy of a hypergraph tells us how well all its hyperedges can be partitioned. Spencer [18] proved a fundamental “six-standard-deviation” result that for any hypergraph $H$ with $n$ vertices and $n$ hyperedges,

$$D(H) \leq 6\sqrt{n}.$$ 

As shown in [1], this bound is best possible up to a constant factor. More precisely, if a Hadamard matrix of order $n > 1$ exists, then there is a hypergraph $H$ with $n$ vertices and $n$ hyperedges such that

$$D(H) \geq 0.5\sqrt{n}.$$ 

It is well known that a Hadamard matrix of order between $n$ and $(1 - \epsilon)n$ does exist for any $\epsilon$ and sufficiently large $n$. The following important result, due to Beck and Fiala [4], is valid for a hypergraph with any number of hyperedges:

$$D(H) \leq 2\Delta - 1,$$

where $\Delta$ is the maximum degree of vertices of $H$. They also posed the discrepancy conjecture that for some constant $K$

$$D(H) < K\sqrt{\Delta}.$$ 

Another interesting aspect of discrepancy was discussed by Füredi and Mubayi in their fundamental paper [9]. A function $g : V \to \{-1, +1\}$ is called a signed domination function (SDF) of the hypergraph $H$ if

$$g(E_i) = \sum_{v \in E_i} g(v) \geq 1$$

for every hyperedge $E_i \in \mathcal{E}$, i.e., each hyperedge has a positive imbalance. The signed discrepancy of $H$, denoted by $SD(H)$, is defined in the following way:

$$SD(H) = \min_{SDF} g(V),$$

where the minimum is taken over all signed domination functions of $H$. Thus, in this version of discrepancy, the success is measured by minimizing the imbalance of the vertex set $V$, while keeping the imbalance of every hyperedge $E_i \in \mathcal{E}$ positive.

One of the main results in this context, formulated in terms of hypergraphs, is due to Füredi and Mubayi [9]:

**Theorem 1 ([9])** Let $H$ be an $n$-vertex hypergraph with hyperedge set $\mathcal{E} = \{E_1, \ldots, E_m\}$, and suppose that every hyperedge has at least $k$ vertices, where $k \geq 100$. Then

$$SD(H) \leq 4\sqrt{\frac{\ln k}{k} n + \frac{1}{k} m}.$$
This theorem can be easily re-formulated in terms of graphs by considering the neighbour- 
hood hypergraph of a given graph. Note that \( \gamma_s(G) \) is the signed domination number 
of a graph \( G \), which is formally defined in the next section.

**Theorem 2 ([9])** If \( G \) has \( n \) vertices and minimum degree \( \delta \geq 99 \), then

\[
\gamma_s(G) \leq \left( 4 \sqrt{\frac{\ln(\delta + 1)}{\delta + 1}} + \frac{1}{\delta + 1} \right)^n.
\]

Moreover, Füredi and Mubayi [9] found quite good upper bounds for very small values of \( \delta \) and, using Hadamard matrices, constructed a \( \delta \)-regular graph \( G \) of order \( 4\delta \) with

\[
\gamma_s(G) \geq 0.5\sqrt{\delta} - O(1).
\]

This construction shows that the upper bound in Theorem 2 is off from optimal by at most the factor of \( \sqrt{\ln \delta} \). They posed an interesting conjecture that, for some constant \( C \),

\[
\gamma_s(G) \leq \frac{C}{\sqrt{\delta}} n,
\]

and proved that the above discrepancy conjecture, if true, would imply this upper bound for \( \delta \)-regular graphs. A strong result of Matoušek [15] shows that the bound is true, but the constant \( C \) in his proof is big making the result of rather theoretical interest.

The lower bound for the signed domination number given in the theorem below is formulated in terms of the degree sequence of a graph. Other lower bounds are also known, see Corollaries 4, 5 and 6.

**Theorem 3 ([7])** Let \( G \) be a graph with degrees \( d_1 \leq d_2 \leq \ldots \leq d_n \). If \( k \) is the smallest integer for which

\[
\sum_{i=0}^{k-1} d_{n-i} \geq 2(n-k) + \sum_{i=1}^{n-k} d_i,
\]

then

\[
\gamma_s(G) \geq 2k - n.
\]

In this paper, we present new upper and lower bounds for the signed domination number, which improve the above theorems and also generalise three known results formulated in Corollaries 4, 5 and 6. Note that our results can be easily re-formulated in terms of hypergraphs. Moreover, we rectify Füredi–Mubayi’s conjecture formulated above as follows: for some \( C \leq 10 \) and \( \alpha, 0.18 \leq \alpha < 0.21 \),

\[
\gamma_s(G) \leq \min \left\{ \frac{n}{\delta^\alpha}, \frac{Cn}{\sqrt{\delta}} \right\}.
\]

# 2 Notation and Technical Results

All graphs will be finite and undirected without loops and multiple edges. If \( G \) is a graph of order \( n \), then \( V(G) = \{v_1, v_2, \ldots, v_n\} \) is the set of vertices in \( G \) and \( d_i \) denotes the degree of \( v_i \). Let \( N(x) \) denote the neighbourhood of a vertex \( x \). Also, let \( N(X) = \bigcup_{x \in X} N(x) \) and \( N[X] = N(X) \cup X \). Denote by \( \delta(G) \) and \( \Delta(G) \) the minimum and maximum degrees of vertices of \( G \), respectively. Put \( \delta = \delta(G) \) and \( \Delta = \Delta(G) \).
A set $X$ is called a dominating set if every vertex not in $X$ is adjacent to a vertex in $X$. The minimum cardinality of a dominating set of $G$ is called the domination number $\gamma(G)$. The domination number can be defined equivalently by means of a domination function, which can be considered as a characteristic function of a dominating set in $G$. A function $f : V(G) \rightarrow \{0,1\}$ is a domination function on a graph $G$ if for each vertex $v \in V(G)$,

$$\sum_{x \in N[v]} f(x) \geq 1. \quad (1)$$

The value $\sum_{v \in V(G)} f(v)$ is called the weight $f(V(G))$ of the function $f$. It is obvious that the minimum of weights, taken over all domination functions on $G$, is the domination number $\gamma(G)$ of $G$.

It is easy to obtain different variations of the domination number by replacing the set $\{0,1\}$ by another set of numbers. If $\{0,1\}$ is exchanged by $\{-1,1\}$, then we obtain the signed domination number. A signed domination function of a graph $G$ was defined in [7] as a function $f : V(G) \rightarrow \{-1,1\}$ such that for each $v \in V(G)$, the expression (1) is true. The signed domination number of a graph $G$, denoted $\gamma_s(G)$, is the minimum of weights $f(V(G))$, taken over all signed domination functions $f$ on $G$. A research on signed domination has been carried out in [7]–[13] and [15].

Let $d \geq 2$ be an integer and $0 \leq p \leq 1$. Let us denote

$$f(d,p) = \sum_{m=0}^{\lfloor 0.5d \rfloor} \left( \lfloor 0.5d \rfloor - m + 1 \right) \left( \begin{array}{c} d + 1 \\ m \end{array} \right) p^m (1-p)^{d+1-m}. \quad \text{B5}$$

We will need the following technical results:

**Lemma 1** ([9]) If $d$ is odd, then

$$f(d+1,p) < 2(1-p)f(d,p).$$

If $d$ is even, then

$$f(d+1,p) < \left( 2p + (1-p) \frac{d+4}{d+2} \right) f(d,p).$$

In particular, if

$$2(1-p) \left( 2p + (1-p) \frac{d+4}{d+2} \right) < 1,$$

then

$$\max_{d \geq \delta} f(d,p) \in \{ f(\delta,p), f(\delta+1,p) \}.$$

**Lemma 2** ([6]) Let $p \in [0,1]$ and $X_1, ..., X_k$ be mutually independent random variables with

$$P[X_i = 1-p] = p,$$
$$P[X_i = -p] = 1 - p.$$ 

If $X = X_1 + ... + X_k$ and $c > 0$, then

$$P[X < -c] < e^{-\frac{c^2}{2pk}}.$$

Let us also denote

$$\tilde{d}_{0.5} = \left( \begin{array}{c} \delta' + 1 \\ \lfloor 0.5\delta' \rfloor \end{array} \right),$$

where

$$\delta' = \left\{ \begin{array}{ll} \delta & \text{if } \delta \text{ is odd;} \\ \delta + 1 & \text{if } \delta \text{ is even.} \end{array} \right.$$
3 Upper Bounds for the Signed Domination Number

The following theorem provides an upper bound for the signed domination number, which is better than the bound of Theorem 2 for 'relatively small' values of δ. For example, if δ(G) = 99, then, by Theorem 2, γₙ(G) ≤ 0.869n, while Theorem 4 yields γₙ(G) ≤ 0.537n. For larger values of δ, the latter result is improved in Corollaries 1–3.

Theorem 4 For any graph G with δ > 1,

\[ γ_δ(G) ≤ \left( 1 - \frac{2^\delta}{(1 + \delta)^{1+1/\delta} \delta^{-1/\delta}} \right) n, \]

where \( \hat{\delta} = [0.5\delta] \).

Proof: Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with the probability

\[ p = 1 - \frac{1}{(1 + \hat{\delta})^{1/\hat{\delta}} \hat{\delta}^{-1/\hat{\delta}}}. \]

For \( m ≥ 0 \), let us denote by \( B_m \) the set of vertices \( v ∈ V(G) \) dominated by exactly \( m \) vertices of \( A \) and such that \( |N[v] ∩ A| < [0.5d_v] + 1 \), i.e.

\[ |N[v] ∩ A| = m ≤ [0.5d_v]. \]

Note that each vertex \( v ∈ V(G) \) is in at most one of the sets \( B_m \) and \( 0 ≤ m ≤ [0.5d_v] \). Then we form a set \( B \) by selecting \( [0.5d_v] - m + 1 \) vertices from \( N[v] \) that are not in \( A \) for each vertex \( v ∈ B_m \) and adding them to \( B \). We construct the set \( D \) as follows: \( D = A ∪ B \). Let us assume that \( f \) is a function \( f : V(G) → \{-1, 1\} \) such that all vertices in \( D \) are labelled by 1 and all other vertices by -1. It is obvious that \( f(V(G)) = |D| - (n - |D|) \) and \( f \) is a signed domination function.

The expectation of \( f(V(G)) \) is

\[ \mathbb{E}[f(V(G))] = 2\mathbb{E}[|D|] - n \]

\[ = 2(\mathbb{E}[|A|] + \mathbb{E}[|B|]) - n \]

\[ ≤ 2 \sum_{i=1}^{n} P(v_i ∈ A) + 2 \sum_{i=1}^{n} \sum_{m=0}^{[0.5d_v]} ([0.5d_v] - m + 1)P(v_i ∈ B_m) - n \]

\[ = 2pm + 2 \sum_{i=1}^{n} \sum_{m=0}^{[0.5d_v]} ([0.5d_v] - m + 1) \binom{d_i + 1}{m} p^m (1 - p)^{d_i + 1 - m} - n \]

\[ ≤ 2pm + 2 \sum_{i=1}^{n} \max_{d_i ≥ \delta} f(d_i, p) - n. \]

It is not difficult to check that \( 2(1 - p)(2p + (1 - p)(d + 4)/(d + 2)) < 1 \) for any \( d ≥ \delta ≥ 2 \). By Lemma 1,

\[ \max_{d_i ≥ \delta} f(d, p) ∈ \{ f(\delta, p), f(\delta + 1, p) \}. \]
The last inequality implies \(2(1 - p) < 1\). Therefore, by Lemma 1,

\[
\max_{d \geq \delta} f(d, p) = f(\delta, p)
\]

if \(\delta\) is odd. If \(\delta\) is even, then we can prove that

\[
\max_{d \geq \delta} f(d, p) = f(\delta + 1, p).
\]

Thus,

\[
\max_{d \geq \delta} f(d, p) = f(\delta', p).
\]

Therefore,

\[
\mathbf{E}[f(V(G))] \leq 2pn + 2n \sum_{m=0}^{[0.5\delta']} \left( [0.5\delta'] - m + 1 \right) \binom{\delta' + 1}{m} p^m (1 - p)^{\delta' + 1 - m} - n.
\]

Since

\[
([0.5\delta'] - m + 1) \binom{\delta' + 1}{m} \leq \binom{\delta' + 1}{[0.5\delta']} \binom{[0.5\delta']}{m},
\]

we obtain

\[
\mathbf{E}[f(V(G))] \leq 2pn + 2n \sum_{m=0}^{[0.5\delta']} \binom{[0.5\delta']}{m} p^m (1 - p)^{[0.5\delta'] - m} - n
\]

\[
= 2pn + 2n \binom{[0.5\delta']}{m} p^m (1 - p)^{[0.5\delta'] - m} - n
\]

\[
= 2pn + 2n \tilde{\delta}_{0.5}(1 - p)^{[0.5\delta'] + 1} - n.
\]

Taking into account that \(\delta' - [0.5\delta'] = [0.5\delta'] = [0.5\delta] = \tilde{\delta}\), we have

\[
\mathbf{E}[f(V(G))] \leq 2pn + 2n \tilde{\delta}_{0.5}(1 - p)^{\tilde{\delta} + 1} - n
\]

\[
\leq \left( 1 - \frac{2 \tilde{\delta}}{(1 + \tilde{\delta})^{1+\delta} \tilde{\delta}_{0.5}} \right) n,
\]

as required. The proof of Theorem 4 is complete.

Our next result and its corollaries give a modest improvement of Theorem 2. More precisely, the upper bound of Theorem 5 is asymptotically 1.63 times better than the bound of Theorem 2, and for \(\delta = 10^6\) the improvement is 1.44 times.

**Theorem 5** If \(\delta(G) \geq 10^6\), then

\[
\gamma_s(G) \leq \frac{\sqrt{6 \ln(\delta + 1)} + 1.21}{\sqrt{\delta + 1}} n.
\]
Proof: Denote \( \delta^+ = \delta + 1 \), \( N_v = N[v] \) and \( n_v = |N_v| \). Let \( A \) be a set formed by an independent choice of vertices of \( G \), where each vertex is selected with the probability

\[
p = 0.5 + \sqrt{1.5 \ln \delta^+ / \delta^+}.
\]

Let us construct two sets \( Q \) and \( U \) in the following way: for each vertex \( v \in V(G) \), if \( |N_v \cap A| \leq 0.5n_v \), then we put \( v \) in \( U \) and add \( [0.5n_v + 1] \) vertices of \( N_v \) to \( Q \). Furthermore, we assign “+” to \( A \cup Q \), and “−” to all other vertices. The resulting function \( g : V(G) \rightarrow \{-1, 1\} \) is a signed domination function, and

\[
g(V(G)) = 2|A \cup Q| - n \leq 2|A| + 2|Q| - n.
\]

The expectation of \( g(V(G)) \) is

\[
E[g(V(G))] \leq 2E[|A|] + 2E[|Q|] - n = 2pn - n + 2E[|Q|].
\]

(3)

It is easy to see that \( |Q| \leq \sum_{v \in U} [0.5n_v + 1] \), hence

\[
E[|Q|] \leq \sum_{v \in V(G)} [0.5n_v + 1] P[v \in U],
\]

(4)

where

\[
P[v \in U] = P[|N_v \cap A| \leq 0.5n_v].
\]

Let us define the following random variables

\[
X_w = \begin{cases} 
1 - p & \text{if } w \in A \\
- p & \text{if } w \notin A 
\end{cases}
\]

and let \( X^*_v = \sum_{w \in N_v} X_w \). We have

\[
|N_v \cap A| \leq 0.5n_v \quad \text{if and only if} \quad X^*_v \leq (1 - p)0.5n_v + (-p)0.5n_v.
\]

Thus,

\[
P[|N_v \cap A| \leq 0.5n_v] = P[X^*_v \leq (0.5 - p)n_v].
\]

By Lemma 2,

\[
P[X^*_v \leq (0.5 - p)n_v] < e^{-\frac{1.5n_v \ln \delta^+ / \delta^+}{1 + \sqrt{6 \ln \delta^+ / \delta^+}}}
\]

For \( n_v \geq \delta^+ > 10^6 \), let us define

\[
y(n_v, \delta^+) = \frac{1.5n_v \ln \delta^+ / \delta^+}{1 + \sqrt{6 \ln \delta^+ / \delta^+}} - \ln(2.25n_v^{1.5}) + 1.
\]

The function \( y(n_v, \delta^+) \) is an increasing function of \( n_v \) and \( y(\delta^+, \delta^+) > 0 \) for \( \delta^+ > 10^6 \). Hence \( y(n_v, \delta^+) \geq y(\delta^+, \delta^+) > 0 \) and

\[
\frac{1.5n_v \ln \delta^+ / \delta^+}{1 + \sqrt{6 \ln \delta^+ / \delta^+}} > \ln(2.25n_v^{1.5}) - 1.
\]

We obtain

\[
P[|N_v \cap A| \leq 0.5n_v] < e^{1 - \ln(2.25n_v^{1.5})} = \frac{e}{2.25n_v^{1.5}},
\]

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and, using inequality (4),

$$2 \mathbb{E}[||Q||] \leq 2 \sum_{v \in V(G)} \frac{e(0.5n_v + 1)}{2.25n_v^{1.5}} \leq \frac{e(\delta + 3)n}{2.25(\delta + 1)^{1.5}} \leq \frac{1.21}{\sqrt{\delta + 1}}n.$$ 

Thus, (3) yields

$$\mathbb{E}[g(V(G))] \leq 2pn - n + \frac{1.21n}{\sqrt{\delta + 1}} = \frac{\sqrt{6 \ln(\delta + 1)} + 1.21}{\sqrt{\delta + 1}}n,$$

as required. The proof of Theorem 5 is complete.

\[ \blacksquare \]

**Corollary 1** If $24,000 \leq \delta$, then

$$\gamma_s(G) \leq \frac{\sqrt{6.8 \ln(\delta + 1)} + 0.32}{\sqrt{\delta + 1}}n.$$ 

**Proof:** Putting $p = 0.5 + \sqrt{1.7 \ln \delta^+ / \delta^+}$ in the proof of Theorem 5, we obtain by Lemma 2,

$$\mathbb{P}[X_v^+ \leq (0.5 - p)n_v] < e^{-\frac{1.7n_v \ln \delta^+ / \delta^+}{1 + \sqrt{6.8 \ln \delta^+ / \delta^+}}}.$$ 

Let us define the following function:

$$y(n_v, \delta^+) = \frac{1.7n_v \ln \delta^+ / \delta^+}{1 + \sqrt{6.8 \ln \delta^+ / \delta^+}} - \ln(3.14n_v^{1.5})$$

for $n_v \geq \delta^+ > 24,000$. The function $y(n_v, \delta^+)$ is an increasing function of $n_v$ and $y(\delta^+, \delta^+) > 0$ for $\delta^+ > 24,000$. Hence $y(n_v, \delta^+) \geq y(\delta^+, \delta^+) > 0$ and

$$\frac{1.7n_v \ln \delta^+ / \delta^+}{1 + \sqrt{6.8 \ln \delta^+ / \delta^+}} > \ln(3.14n_v^{1.5}).$$

We obtain

$$2 \mathbb{E}[||Q||] \leq 2 \sum_{v \in V(G)} \frac{0.5n_v + 1}{3.14n_v^{1.5}} \leq \frac{(\delta + 3)n}{3.14(\delta + 1)^{1.5}} \leq \frac{0.32}{\sqrt{\delta + 1}}n.$$ 

Thus, (3) yields

$$\mathbb{E}[g(V(G))] \leq 2pn - n + \frac{0.32n}{\sqrt{\delta + 1}} = \frac{\sqrt{6.8 \ln(\delta + 1)} + 0.32}{\sqrt{\delta + 1}}n,$$

as required. The proof is complete.

\[ \blacksquare \]

**Corollary 2** If $1,000 \leq \delta \leq 24,000$, then

$$\gamma_s(G) \leq \frac{\sqrt{\ln(\delta + 1)(11.8 - 0.48 \ln \delta) + 0.25}}{\sqrt{\delta + 1}}n.$$
Proof: It is similar to the proof of Corollary 1 if we put

\[ p = 0.5 + \sqrt{(2.95 - 0.12 \ln \delta) \ln \delta^+ / \delta^+} \]

and consider the following function for \(1,000 \leq \delta \leq 24,000\):

\[ y(n_v, \delta^+) = \frac{(2.95 - 0.12 \ln \delta)n_v \ln \delta^+ / \delta^+}{1 + \sqrt{(11.8 - 0.48 \ln \delta) \ln \delta^+ / \delta^+}} - \ln(4.01n_v^{1.5}). \]

\[ \Box \]

Corollary 3 If \(230 \leq \delta \leq 1,000\), then

\[ \gamma_s(G) \leq \frac{\sqrt{\ln(\delta + 1)(18.16 - 1.4 \ln \delta)} + 0.25}{\sqrt{\delta + 1}}n. \]

Proof: It is similar to the proof of Corollary 1 if we put

\[ p = 0.5 + \sqrt{(4.54 - 0.35 \ln \delta) \ln \delta^+ / \delta^+} \]

and consider the following function for \(230 \leq \delta \leq 1,000\):

\[ y(n_v, \delta^+) = \frac{(4.54 - 0.35 \ln \delta)n_v \ln \delta^+ / \delta^+}{1 + \sqrt{(18.16 - 1.4 \ln \delta) \ln \delta^+ / \delta^+}} - \ln(4.04n_v^{1.5}). \]

We believe that Füredi–Mubayi’s conjecture, saying that \(\gamma_s(G) \leq \frac{Cn}{\sqrt{\delta}}\), is true for some small constant \(C\). However, as the Peterson graph shows, \(C > 1\), i.e. the behaviour of the conjecture is not good for relatively small values of \(\delta\). Therefore, we propose the following rectified conjecture, which, roughly speaking, consists of two functions for ‘small’ and ‘large’ values of \(\delta\).

Conjecture 1 For some \(C \leq 10\) and \(\alpha, 0.18 \leq \alpha < 0.21\),

\[ \gamma_s(G) \leq \min \left\{ \frac{n}{\delta^{\alpha}}, \frac{Cn}{\sqrt{\delta}} \right\}. \]

The above results imply that if \(C = 10\) and \(\alpha = 0.13\), then this upper bound is true for all graphs with \(\delta \leq 96 \times 10^4\).

4 A Lower Bound for the Signed Domination Number

The following theorem provides a lower bound for the signed domination number of a graph \(G\) depending on its order and a parameter \(\lambda\), which is determined on the basis of the degree sequence of \(G\) (note that \(\lambda\) may be equal to 0, in this case we put \(\sum_{i=1}^{\lambda} = 0\)). This result improves the bound of Theorem 3 and, in some cases, it provides a much better lower bound. For example, let us consider a graph \(G\) consisting of two vertices of degree 5 and \(n - 2\) vertices of degree 3. Then, by Theorem 3,

\[ \gamma_s(G) \geq 0.25n - 1, \]

while Theorem 6 yields

\[ \gamma_s(G) \geq 0.5n - 1. \]
Theorem 6 Let $G$ be a graph with $n$ vertices and degrees $d_1 \leq d_2 \leq \ldots \leq d_n$. Then

$$\gamma_s(G) \geq n - 2\lambda,$$

where $\lambda \geq 0$ is the largest integer such that

$$\sum_{i=1}^{\lambda} \left\lfloor \frac{d_i}{2} + 1 \right\rfloor \leq \sum_{i=\lambda+1}^{n} \left\lfloor \frac{d_i}{2} \right\rfloor.$$

Proof: Let $f$ be a signed domination function of minimum weight of the graph $G$. Let us denote

$$X = \{ v \in V(G) : f(v) = -1 \},$$

and

$$Y = \{ v \in V(G) : f(v) = 1 \}.$$

We have

$$\gamma_s(G) = f(V(G)) = |Y| - |X| = n - 2|X|.$$ 

By definition, for any vertex $v$,

$$f[v] = \sum_{u \in N[v]} f(u) \geq 1.$$

Therefore, for all $v \in V(G)$,

$$|N[v] \cap Y| - |N[v] \cap X| \geq 1.$$

Using this inequality, we obtain

$$|N[v]| = \deg(v) + 1 = |N[v] \cap Y| + |N[v] \cap X| \leq 2|N[v] \cap Y| - 1.$$

Hence

$$|N[v] \cap Y| \geq \frac{\deg(v)}{2} + 1.$$

Since $|N[v] \cap Y|$ is an integer, we conclude

$$|N[v] \cap Y| \geq \left\lceil \frac{\deg(v)}{2} \right\rceil + 1$$

and

$$|N[v] \cap X| = \deg(v) + 1 - |N[v] \cap Y| \leq \left\lfloor \frac{\deg(v)}{2} \right\rfloor.$$

Denote by $e_{X,Y}$ the number of edges between the parts $X$ and $Y$. We have

$$e_{X,Y} = \sum_{v \in X} |N[v] \cap Y| \geq \sum_{v \in X} \left( \left\lceil \frac{\deg(v)}{2} \right\rceil + 1 \right) \geq \sum_{i=1}^{\lfloor |X|/2 \rfloor} \left( \left\lceil \frac{d_i}{2} \right\rceil + 1 \right).$$

Note that if $X = \emptyset$, then we put $\sum_{i=1}^{0} g(i) = 0$. On the other hand,

$$e_{X,Y} = \sum_{v \in Y} |N[v] \cap X| \leq \sum_{v \in Y} \left\lfloor \frac{\deg(v)}{2} \right\rfloor \leq \sum_{i=n-|Y|+1}^{\lfloor |Y|/2 \rfloor} d_i / 2 = \sum_{i=\lfloor |Y|/2 \rfloor + 1}^{n} d_i / 2.$$
Therefore, the following inequality holds:
\[
\sum_{i=1}^{\lvert X \rvert} \left( \left\lceil \frac{d_i}{2} \right\rceil + 1 \right) \leq \sum_{i=1}^{n} \left\lfloor \frac{d_i}{2} \right\rfloor.
\]
Since \( \lambda \geq 0 \) is the largest integer such that
\[
\sum_{i=1}^{\lambda} \left( \left\lceil \frac{d_i}{2} \right\rceil + 1 \right) \leq \sum_{i=\lambda+1}^{n} \left\lfloor \frac{d_i}{2} \right\rfloor,
\]
we conclude that
\[
\lvert X \rvert \leq \lambda.
\]
Thus,
\[
\gamma_s(G) = n - 2\lvert X \rvert \geq n - 2\lambda.
\]
The proof is complete.

Theorem 6 immediately implies the following known results:

**Corollary 4 ([10] and [19])** For any graph \( G \),
\[
\gamma_s(G) \geq \left( \left\lfloor 0.5\delta \right\rfloor - \left\lceil 0.5\Delta \right\rceil + 1 \right) n.
\]

Note that Haas and Wexler [10] formulated the above bound only for graphs with \( \delta \geq 2 \), while Zhang et al. [19] proved a weaker version without the ceiling and floor functions.

**Corollary 5 ([13])** If \( \delta \) is odd and \( G \) is \( \delta \)-regular, then
\[
\gamma_s(G) \geq \frac{2n}{\delta + 1}.
\]

**Corollary 6 ([7])** If \( \delta \) is even and \( G \) is \( \delta \)-regular, then
\[
\gamma_s(G) \geq \frac{n}{\delta + 1}.
\]

Disjoint unions of complete graphs show that these lower bounds are sharp whenever \( n/(\delta + 1) \) is an integer, and therefore the bound of Theorem 6 is sharp for regular graphs.

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