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Perfect graphs of strong domination and independent strong domination

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Abstract

Let $\gamma(G)$, $i(G)$, $\gamma_S(G)$ and $i_S(G)$ denote the domination number, the independent domination number, the strong domination number and the independent strong domination number of a graph G, respectively. A graph G is called *γi-perfect* (*domination perfect*) if $\gamma(H) = i(H)$, for every induced subgraph H of G. The classes of $\gamma\gamma_S$-perfect, $\gamma_S i_S$-perfect, ii_S-perfect and γi_S-perfect graphs are defined analogously. In this paper we present a number of characterization results on the above classes of graphs. For example, characterizations of K_4-free $\gamma_S i_S$-perfect graphs and triangle-free γi_S-perfect graphs are given. Moreover, the strong dominating set and independent strong dominating set problems as well as the weak dominating set and independent weak dominating set problems are shown to be NP-complete on a class of graphs. Several problems and conjectures are proposed. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

All graphs will be finite and undirected, without loops and multiple edges. If G is a graph, $V(G)$ denotes the set of vertices in G. For $X, Y \subseteq V(G)$, we write $X \perp Y$ if any vertex of X is adjacent to all vertices of the set Y, and $X \pm Y$ if no vertex of X is adjacent to any vertex of Y. In particular, $x \perp y$ means that the vertices x and y are adjacent. Let $N(x)$ denote the neighborhood of a vertex x, and let $G[X]$ denote the subgraph of G induced by $X \subseteq V(G)$. Also, let $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. The degree of a vertex u in G is denoted by $d(u)$.

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A set $X \in V(G)$ *dominates* a set $Y \in V(G)$ if $Y \subseteq N[X]$. In particular, if X dominates $V(G)$, then X is called a *dominating set*. The *independent domination number* $i(G)$ is the cardinality of a minimum independent dominating set of G, and the *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of G. A set X is a *strong dominating set* if every vertex $u \in V(G) - X$ is adjacent to a vertex $v \in X$ with $d(v) \geqslant d(u)$. The *strong domination number* $\gamma_{\mathrm{S}}(G)$ is the minimum cardinality of a strong dominating set of G, and the *independent strong domination number* $i_{\mathrm{S}}(G)$ is the minimum cardinality of an independent strong dominating set of G. Since a greedy algorithm produces an independent strong dominating set, the parameter $i_{\mathrm{S}}(G)$ is well defined. The strong domination and independent strong domination numbers were studied in [6,9,13,14,16]. It follows from the definitions that for any graph G,

$$\gamma(G) \leqslant i(G) \leqslant i_{\mathrm{S}}(G)$$

and

$$\gamma(G) \leqslant \gamma_{\mathrm{S}}(G) \leqslant i_{\mathrm{S}}(G).$$

Sumner and Moore [17] define a graph G to be *domination perfect* if $\gamma(H) = i(H)$, for every induced subgraphs H of G. A summary of known results on domination perfect graphs and their characterization in terms of 17 forbidden-induced subgraphs can be found in [19]. Similarly, a graph G is called $\gamma_{\mathrm{S}} i_{\mathrm{S}}$*-perfect* ($\gamma\gamma_{\mathrm{S}}$*-perfect*, ii_{S}*-perfect*, γi_{S}*-perfect*) if $\gamma_{\mathrm{S}}(H) = i_{\mathrm{S}}(H)$ ($\gamma(H) = \gamma_{\mathrm{S}}(H)$, $i(H) = i_{\mathrm{S}}(H)$, $\gamma(H) = i_{\mathrm{S}}(H)$, resp.), for every induced subgraph H of G. The class of $\gamma_{\mathrm{S}} i_{\mathrm{S}}$-perfect graphs is a direct analogue of domination perfect graphs and it will be referred to as strong domination perfect graphs. Besides domination perfect graphs, many other analogous classes of graphs have been studied. For example, neighborhood perfect graphs, irredundance perfect graphs, upper domination and upper irredundance perfect graphs, $\alpha\alpha'$-perfect graphs, where α is either the Grundy number or the achromatic number while α' stands for either the clique number or the chromatic number, are known — we provide just a few Refs. [2–5,10,12,18,20].

In this paper we show that strong domination perfect graphs form a subclass of the well-known class of domination perfect graphs. We also present a sufficient condition in terms of forbidden-induced subgraphs for a graph to be strong domination perfect implying the known result that any $K_{1,3}$-free graph is strong domination perfect as well as characterizations of K_4-free strong domination perfect graphs and C_4-free strong domination perfect graphs. A characterization of triangle-free γi_{S}-perfect graphs and characterizations of $\{C_3, C_5\}$-free $\gamma\gamma_{\mathrm{S}}$-perfect graphs and ii_{S}-perfect graphs are given. Moreover, the strong dominating set and independent strong dominating set problems as well as the weak dominating set and independent weak dominating set problems are shown to be NP-complete on the class of bipartite planar graphs of maximum degree 3 and with girth at least k for a fixed k.

2. Strong domination perfect graphs

The following theorem shows that strong domination perfect graphs form a subclass of the well-known class of domination perfect graphs. We can prove this theorem

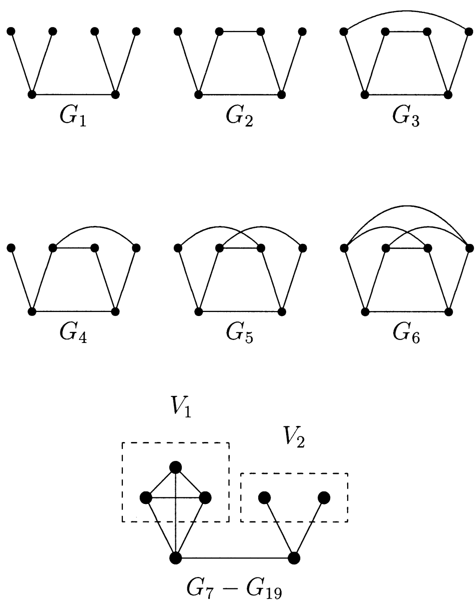

Fig. 1.

using a characterization of domination perfect graphs in terms of 17 forbidden induced subgraphs [19]. Instead, we provide a short direct proof of the result.

Theorem 1. *Any strong domination perfect graph is domination perfect.*

Proof. Let G be a strong domination perfect graph. We suppose that G is not domination perfect, i.e. $\gamma(H) < i(H)$ for some induced subgraph H of G. We choose a minimum dominating set D of H such that the number of edges in the graph $H[D]$ is minimum. Since $\gamma(H) < i(H)$, there is an edge $uv \in E(H[D])$. We define

$$\mathrm{PN}(u) = \{x \in V(H) - D \mid N(x) \cap D = \{u\}\}$$

and

$$\mathrm{PN}(v) = \{x \in V(H) - D \mid N(x) \cap D = \{v\}\}.$$

The set $\mathrm{PN}(u)$ is not empty, since D is a minimum dominating set, and hence there is a vertex $a \in \mathrm{PN}(u)$. Furthermore, suppose that the graph $H[\mathrm{PN}(u)]$ is complete. We obtain that the set $(D - \{u\}) \cup \{a\}$ is a minimum dominating set of H containing fewer edges than D which contradicts the choice of D. Thus, the graph $H[\mathrm{PN}(u)]$ is not complete and hence there are two non-adjacent vertices $a, a' \in \mathrm{PN}(u)$. We can show analogously that there are also two non-adjacent vertices $b, b' \in \mathrm{PN}(v)$. Now, any of the graphs $G[u, v, a, b, a', b']$ is isomorphic to one of the graphs G_1–G_6 in Fig. 1.

We obtain

$$\gamma_S(G_i) = 2 < i_S(G_i) = 3$$

for $i = 1,\dots,6$, contrary to the fact that G is strong domination perfect. □

A sufficient condition for a graph to be strong domination perfect is given in the next theorem. The graphs G_7–G_{19} in Fig. 1 are the 13 non-isomorphic graphs which arise by adding any combination of edges between V_1 and V_2.

Theorem 2. *If a graph G does not contain any of the graphs G_1–G_{19} shown in Fig.* 1 *as an induced subgraph, then G is a strong domination perfect graph.*

Proof. We assume that G is a minimal counterexample, i.e. G does not contain any of the graphs $G_1,\dots,G_{19}$ as an induced subgraph and $\gamma_S(G) < i_S(G)$. We choose a minimum strong dominating set D according to the following conditions:

1. Let $E' = E(G[D])$. The value of

   $$\max\{d(x) + d(y) \mid xy \in E'\}$$

   is minimized. We define this value to be t.
2. Subject to Condition 1, the number of edges xy in E' with $d(x) + d(y) = t$ is minimized.

Since $\gamma_S(G) < i_S(G)$, we have $t > 0$ and $E' \neq \emptyset$. For every vertex $y \in D$ adjacent to a vertex $x \in D$ with $d(x) \geqslant d(y)$, there is at least one vertex in $V(G) - D$ which is strongly dominated only by $y \in D$, since D is a minimal strong dominating set. The set of all these vertices is denoted by N_y.

Claim 1. *If $x, y \in D$ with $xy \in E'$, $d(x) \geqslant d(y)$ and $d(x) + d(y) = t$, then $N(x) \cap N_y = \emptyset$ and $G[N_y]$ is not complete.*

Proof. If $w \in N(x)$ for some $w \in N_y$, then x strongly dominates w which contradicts the choice of N_y. Suppose $G[N_y]$ is complete. We choose $v \in N_y$ with $d(v) = \max\{d(u) \mid u \in N_y\}$ and consider the set $D' = (D - \{y\}) \cup \{v\}$. Clearly, D' is a minimum strong dominating set. We set $E'' = E(G[D'])$. Suppose there is an edge $vz \in E'' - E'$ for some $z \in D$. Since $v \in N_y$, we have $d(v) > d(z)$. Hence,

$$d(v) + d(z) < d(y) + d(v) \leqslant d(x) + d(y) = t$$

and, therefore,

$$|\{ab \in E'' \mid d(a) + d(b) = t\}| < |\{ab \in E' \mid d(a) + d(b) = t\}|,$$

which is a contradiction to the choice of D. This completes the proof of the claim. □

We proceed in the proof of the theorem and choose $xy \in E'$ with $d(x) \geqslant d(y)$ and $d(x) + d(y) = t$. By the claim, we know that N_y contains two non-adjacent vertices

a, b such that $a, b \notin N(x)$. If $d(x) = d(y)$, then, by the claim, also N_x contains two non-adjacent vertices c, d such that $c, d \notin N(y)$. Now, the graph $G[x, y, a, b, c, d]$ is one of the graphs $G_1, \ldots, G_6$, which is a contradiction.

Hence, $d(x) > d(y)$. We set $k = |N(x) \cap N(y)|$ and $l = |N(x) - N[y]|$. Then $d(x) = k + l + 1$ and $d(y) \geqslant k + 1 + |N_y| \geqslant k + 3$. Now $d(x) > d(y)$ implies $l \geqslant 3$. If there are two non-adjacent vertices $c, d \in N(x) - N[y]$, then $G[x, y, a, b, c, d]$ is one of the graphs $G_1, \ldots, G_6$, which is a contradiction. Therefore, $G[\{x\} \cup (N(x) - N[y])]$ is a complete graph of order at least $1 + l \geqslant 4$. Now, for $c, d, e \in N(x) - N[y]$ the graph $G[x, y, a, b, c, d, e]$ is one of the graphs $G_7, \ldots, G_{19}$, which is a contradiction and the proof is complete. □

Corollary 1 (Sampathkumar and Pushpa Latha [16]). *Any $K_{1,3}$-free graph is strong domination perfect.*

Proof. This follows immediately from Theorem 2, since all the graphs G_1–G_{19} contain $K_{1,3}$ as an induced subgraph. □

Corollary 2 (Allan and Laskar [1]). *Any $K_{1,3}$-free graph is domination perfect.*

Proof. This follows immediately from Theorem 1 and Corollary 1. □

Corollary 3. *A K_4-free graph is strong domination perfect if and only if it does not contain any of the graphs G_1–G_6 in Fig.* 1 *as an induced subgraph.*

Proof. By Theorem 2, we only have to prove the 'only if'-part which is immediate, since $\gamma_S(G_i) < i_S(G_i)$ for $i = 1, \ldots, 6$. □

Let G_7 denote the graph in Fig. 1 for which $E(G_7[V_1, V_2]) = \emptyset$.

Corollary 4. *A C_4-free graph is strong domination perfect if and only if it does not contain any of the graphs G_1, G_7 in Fig.* 1 *as an induced subgraph.*

Proof. This follows immediately from Theorem 2, since all the graphs G_2–G_6, G_8–G_{19} *contain C_4 as an induced subgraph and $\gamma_S(G_i) < i_S(G_i)$ for $i = 1, 7$.* □

Notice that Corollary 3 implies characterizations of triangle-free strong domination perfect graphs and bipartite strong domination perfect graphs, while Corollary 4 implies a characterization of chordal strong domination perfect graphs.

3. $\gamma\gamma_S$-perfect and γi_S-perfect graphs

In this section we deal with $\gamma\gamma_S$-perfect graphs and γi_S-perfect graphs. A characterization of $\gamma\gamma_S$-perfect graphs containing neither C_3 nor C_5 as an induced subgraph is given in the next theorem.

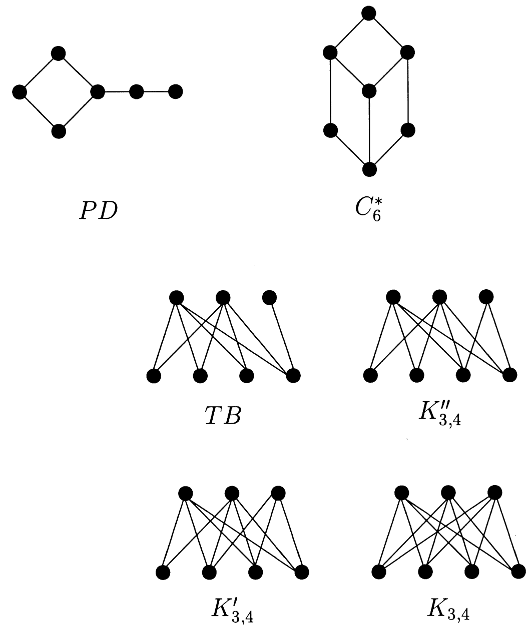

Fig. 2.

Theorem 3. *A $\{C_3, C_5\}$-free graph G is $\gamma\gamma_S$-perfect if and only if it does not contain any of the 7 graphs of Fig. 2 as an induced subgraph.*

Proof. The 'only if'-part is trivial because all the graphs G of Fig. 2 satisfy $\gamma(G) < \gamma_S(G)$. Hence, we only prove the 'if'-part. Assume that G is a minimal counterexample, i.e. G is a graph of minimal order such that $\gamma(G) < \gamma_S(G)$, G is $\{C_3, C_5\}$-free and G does not contain any of the graphs in Fig. 2 as an induced subgraph. We choose a minimum dominating set D of G such that

$$d(D) = \sum_{x \in D} d(x)$$

is maximum. As $\gamma(G) < \gamma_S(G)$, D is not a strong dominating set of G and hence there is a vertex $v \in V(G) - D$ that has no strong neighbor in D, i.e. $d(v) > d(x)$ for every $x \in D \cap N(v)$. If $d(v) = 2$, then the set $D' = (D - \{x\}) \cup \{v\}$ for some $x \in N(v) \cap D$ is a dominating set with $d(D') > d(D)$, which is a contradiction. Hence, we can assume that $d(v) \geqslant 3$. We define

$$N_0 = N(v) \cap D = \{x_1, \ldots, x_l\}, \quad l \geqslant 1,$$

$$N_1 = N(v) \cap (V(G) - D) = \{a_1, \ldots, a_k\}$$

with $k \geqslant 0$ and $k + l = d(v)$, and

$$N_2 = N(N_1) \cap D = \{b_1, \ldots, b_p\}.$$

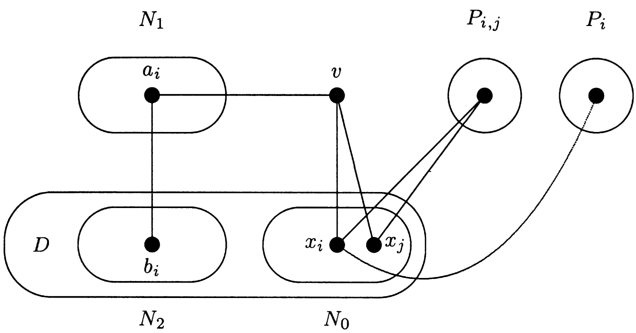

Fig. 3.

Since D is a dominating set, $N_1 \subseteq N(N_2)$. Furthermore, for $1 \leqslant i \leqslant j \leqslant l$, we define

$$P_{i,j} = (N(x_i) \cap N(x_j)) \cap (V(G) - (D \cup \{v\})) - N(D - \{x_i, x_j\})$$

and $P_i = P_{i,i}$. Since G is $\{C_3, C_5\}$-free, the set $N_0 \cup N_1$ is independent and the following set is independent as well:

$$\bigcup_{\substack{i,j=1\\ i \leqslant j}}^{l} P_{i,j} \cup N_2 \cup \{v\}.$$

Note that $P_i \neq \emptyset$ for $1 \leqslant i \leqslant l$, for otherwise $D' = (D - \{x_i\}) \cup \{v\}$ would be a minimum dominating set of G with $d(D') > d(D)$. Suppose that $|N_0| \geqslant 3$. Choosing $y_i \in P_i$ for $i = 1, 2, 3$ we obtain $G[v, x_1, y_1, x_2, y_2, x_3, y_3] \cong T^*$, a contradiction (see Fig. 3).

Consider the case $|N_0| = 2$. First, we assume that there is a vertex $a \in N_1$ with $a \perp y$ for some vertex $y \in P_1 \cup P_2$. We assume, w.l.o.g., $y \in P_1$. If there is a $y' \in N(x_2) - N(x_1) - N(a)$, then $G[v, a, x_1, y, x_2, y'] \cong \mathrm{PD}$. Hence $P_2 \subseteq N(x_2) - N(x_1) \subseteq N(a)$ and, by symmetry, $P_1 \subseteq N(x_1) - N(x_2) \subseteq N(a)$. If there is a $y' \in (N(x_1) \cap N(x_2)) - N(a)$, then for $y_i \in P_i$ $(i = 1, 2)$ we get $G[v, a, x_1, x_2, y_1, y_2, y'] \cong C_6^*$. Hence $P_{1,2} \subseteq N(x_1) \cap N(x_2) \subseteq N(a)$. We obtain

$$d(a) \geqslant |N(x_2) - N(x_1)| + |N(x_1) \cap N(x_2)| = d(x_2)$$

and the set $D' = (D - \{x_1, x_2\}) \cup \{a, v\}$ is a minimum dominating set of G with $d(D') > d(D)$, a contradiction. Hence, $N_1 \cup P_1 \cup P_2$ is independent. We choose $y_i \in P_i$ for $i = 1, 2$. If $b \pm x_i$ for $i = 1, 2$ and a vertex $b \in N_2$, then $G[v, a, b, x_1, y_1, x_2, y_2] \cong T^*$ for $a \in N(b) \cap N_1$. If $b \perp x_1$ and $b \pm x_2$, then $G[v, a, b, x_1, x_2, y_2] \cong \mathrm{PD}$. Hence, $N_2 \perp N_0 = \{x_1, x_2\}$. If there is a vertex $b \in N_2$ with $|N(b) \cap N_1| \geqslant 2$, then $G[b, a_1, a_2, v, x_1, x_2, y_1] \cong \mathrm{TB}$ for $a_1, a_2 \in N(b) \cap N_1$. Hence, $|N(b) \cap N_1| = 1$ for all $b \in N_2$ which implies $|N_2| \geqslant |N_1|$ since $N_1 \subseteq N(N_2)$. Now, we have $d(x_1) \geqslant 1 + |P_1| + |N_2| \geqslant 2 + |N_1| = d(v)$, a contradiction.

Consequently, $|N_0| = 1$. This implies $|N_1| \geqslant 2$. We consider two cases.

Case 1: $N_0 \cup N_2$ *is independent*. Suppose that there is a vertex $b \in N_2$ with $|N(b) \cap N_1| \geqslant 2$. We choose $a_1, a_2 \in N(b) \cap N_1$, $x_1 \in N_0$ and $y \in P_1$. If $y \pm a_i$ for $i = 1, 2$, then $G[v, x_1, y, a_1, a_2, b] \cong \mathrm{PD}$ and hence $y \perp a_1$ or $y \perp a_2$ for every $y \in P_1$. If there are two vertices $y_1, y_2 \in P_1$ with $y_1 \perp a_1$, $y_1 \pm a_2$, $y_2 \pm a_1$, $y_2 \perp a_2$, then $G[v, x_1, y_1, y_2, a_1, a_2, b] \cong C_6^*$ and hence every pair $y_1, y_2 \in P_1$ has a common neighbor in $\{a_1, a_2\}$. Inductively, this implies that all vertices in P_1 have a common neighbor in $\{a_1, a_2\}$. We assume w.l.o.g. that $a_1 \perp P_1$. If there is a $c \in N(b) - N(v)$, then $c \pm x_1$, for otherwise we have C_5. If there is a vertex $y \in P_1$ with $c \pm y$, then $G[v, x_1, y, a_1, b, c] \cong$ PD and hence $c \perp P_1$. We obtain $G[v, x_1, y, a_1, a_2, b, c] \cong C_6^*$ and hence $N(b) \subseteq N(v)$. If $d(b) > d(a_1) \geqslant 3$, then there are two other vertices $a_3, a_4 \in N(b) \cap N(v)$. As above, we obtain $y \perp a_i$ or $y \perp a_j$ for $1 \leqslant i < j \leqslant 4$ and $y \in P_1$ which implies that $|N(y) \cap \{a_1, a_2, a_3, a_4\}| \in \{3, 4\}$. We assume w.l.o.g. that $y \perp \{a_1, a_2, a_3\}$. Then $G[v, a_1, a_2, a_3, a_4, b, y] \cong K'_{3,4}$ or $K_{3,4}$ and hence $d(b) \leqslant d(a_1)$. Thus, $D' = (D - \{x_1, b\}) \cup \{v, a_1\}$ is a dominating set of G with $d(D') > d(D)$, a contradiction.

Therefore, $|N(b) \cap N_1| = 1$ for every $b \in N_2$. We choose $a_1, a_2 \in N_1$, $b_i \in N_2 \cap N(a_i)$ for $i = 1, 2$, and $y \in P_1$. If $y \pm \{a_1, a_2\}$, then $G[v, x_1, y, a_1, b_1, a_2, b_2] \cong T^*$. If $y \perp a_1$ and $y \pm a_2$, then $G[v, x_1, y, a_1, a_2, b_2] \cong$ PD and hence $y \perp \{a_1, a_2\}$. Assume that there is a $c \in N(b_1) - N(v)$. If $y \pm c$, then $G[v, x_1, y, a_1, b_1, c] \cong$ PD and hence $y \perp c$. We know that $b_1 \pm a_2$. If $b_2 \pm c$, then $G[b_1, c, y, a_1, a_2, b_2] \cong$ PD and hence $b_2 \perp c$. Thus, $G[c, b_1, b_2, y, a_1, a_2, v] \cong C_6^*$, a contradiction, and hence $N(b_1) \subseteq N(v)$. Since $N_0 \cup N_2$ is independent, we get $d(b_1) = 1$ and the set $D' = (D - \{b_1\}) \cup \{a_1\}$ is a minimum dominating set with $d(D') > d(D)$, a contradiction.

Case 2: $b \perp x_1$ *for some* $b \in N_2$. We choose $b_1 \in N_2$ such that

$$|N(b_1) \cap N_1| = \max\{|N(b) \cap N_1| \mid b \in N_2 \cap N(x_1)\}.$$

Also, we choose $y_1 \in P_1$. If $a_1, a_2, a_3 \in N(b_1) \cap N_1$, then $G[v, x_1, y_1, b_1, a_1, a_2, a_3] \cong$ TB, $K'_{3,4}$, $K''_{3,4}$ or $K_{3,4}$, a contradiction.

Suppose that $|N(b_1) \cap N_1| = 2$. We choose $a_0, a_1 \in N(b_1) \cap N_1$ and (only for this case) we change the indices in N_1 such that $N_1 = \{a_0, a_1, \ldots, a_{d(v)-2}\}$. We have $N_1 - \{a_0, a_1\} \neq \emptyset$, as $d(v) > d(x_1) \geqslant 3$. For $2 \leqslant i \leqslant d(v) - 2$ there is a vertex $b_i \in N(a_i) \cap N_2$ with $b_1 \neq b_i$, since $b_1 \pm a_i$. Assume that $b_i \pm x_1$ for some $2 \leqslant i \leqslant d(v) - 2$. We have $b_i \perp a_0$, for otherwise $G[b_1, a_0, v, x_1, a_i, b_i] \cong$ PD, and also $b_i \perp a_1$, for otherwise $G[b_1, a_1, v, x_1, a_i, b_i] \cong$ PD. Suppose that there is a $c \in N(b_1) - N(v)$. Since C_3 and C_5 are forbidden, $c \pm \{a_0, a_i\}$. We obtain $c \perp b_i$, for otherwise $G[b_i, a_i, v, a_0, b_1, c] \cong$ PD. Thus, $N[b_1] \subseteq N(\{b_i, x_1\})$ and hence $D - \{b_1\}$ is a dominating set, a contradiction. Hence, $b_i \perp x_1$ for $1 \leqslant i \leqslant d(v) - 2$. If $b_i \pm \{a_0, a_1\}$ for some $2 \leqslant i \leqslant d(v) - 2$, then $G[b_1, a_0, a_1, v, a_i, b_i] \cong$ PD and thus $b_i \perp a_0$ or $b_i \perp a_1$. This implies $N(b_i) \cap (N_1 - \{a_0, a_1\}) = \{a_i\}$ for $2 \leqslant i \leqslant d(v) - 2$ and $b_i \neq b_j$ for $1 \leqslant i < j \leqslant d(v) - 2$. Therefore,

$$\begin{aligned} d(v) &= 1 + |N_1| = 1 + |\{a_0, a_1, a_2, \ldots, a_{d(v)-2}\}| \\ &= 1 + 1 + |\{b_1, b_2, \ldots, b_{d(v)-2}\}| \\ &\leqslant d(x), \end{aligned}$$

which is a contradiction.

At last, let $N(b_1) \cap N_1 = \{a_1\}$. We have $N_1 = \{a_1, a_2, \ldots, a_{d(v)-1}\}$, $b_1 \pm a_i$ for $2 \leqslant i \leqslant d(v) - 1$ and $d(v) - 1 \geqslant d(x_1) \geqslant 3$. Therefore, there is a vertex $b_i \in N(a_i) \cap N_2$

with $b_1 \neq b_i$ for $2 \leqslant i \leqslant d(v)-1$. Assume that $b_i \pm x_1$ for some $2 \leqslant i \leqslant d(v)-1$. If $b_i \pm a_1$, then $G[b_1, a_1, v, x_1, a_i, b_i] \cong \mathrm{PD}$ and thus $b_i \perp a_1$. Since D is a minimum dominating set, there is a $c \in N(b_1) - N[D - \{b_1\}]$. We have $c \pm v$ and also $c \pm \{a_1, a_i\}$, since G is $\{C_3, C_5\}$-free. Now $G[b_i, a_i, v, a_1, b_1, c] \cong \mathrm{PD}$, a contradiction. Hence, $b_i \perp x_1$ for every $1 \leqslant i \leqslant d(v) - 1$ which implies $N(b_i) \cap N_1 = \{a_i\}$ and $b_i \neq b_j$ for $1 \leqslant i < j \leqslant d(v) - 1$. We obtain

$$\begin{aligned} d(v) &= 1 + |N_1| = 1 + |\{a_1, a_2, a_3, \ldots a_{d(v)-1}\}| \\ &= 1 + |\{b_1, b_2, \ldots, b_{d(v)-1}\}| \\ &\leqslant d(x), \end{aligned}$$

which is a contradiction. The proof of the theorem is complete. □

The next corollaries follow directly from Theorem 3.

Corollary 5. *A bipartite graph is $\gamma\gamma_S$-perfect if and only if it does not contain any of the graphs of Fig. 2 as an induced subgraph.*

Corollary 6. *A graph G with girth $g(G) > 5$ is $\gamma\gamma_S$-perfect if and only if it does not contain T^* in Fig. 2 as an induced subgraph.*

Now we consider γi_S-perfect graphs. Theorem 2 from Section 2 enables us to characterize triangle-free γi_S-perfect graphs.

Theorem 4. *A triangle-free graph G is γi_S-perfect if and only if it does not contain any of the graphs G_1–G_6 in Fig. 1 and PD, T^* in Fig. 2 as an induced subgraph.*

Proof. The 'only if'-part follows from the fact that $\gamma < i_S$ for any of the forbidden graphs. To prove the 'if'-part, let G be a minimum counterexample, i.e. G is a graph of minimal order such that $\gamma(G) < i_S(G)$, G is triangle-free and it does not contain any of the above graphs as induced subgraphs. By Theorem 2, $\gamma_S(G) = i_S(G)$ and therefore $\gamma(G) < \gamma_S(G)$. We choose a minimum dominating set D of G such that

$$d(D) = \sum_{x \in D} d(x)$$

is maximum. As $\gamma(G) < \gamma_S(G)$, D is not a strong dominating set of G and hence there is a vertex $v \in V(G) - D$ that has no strong neighbor in D, i.e. $d(v) > d(x)$ for every $x \in D \cap N(v)$. If $d(v) = 2$, then the set $D' = (D - \{x\}) \cup \{v\}$ for some $x \in N(v) \cap D$ is a dominating set with $d(D') > d(D)$, which is a contradiction. Hence, we can assume that $d(v) \geqslant 3$.

Claim 1. *If $u \in V(G)$ and $d(u) \geqslant 3$, then $d(w) \leqslant 2$ for any vertex $w \in N(u)$.*

Proof. Suppose to the contrary that there are adjacent vertices u, w of degree at least three. Since G is triangle-free, the set $N(u) - N[w]$ contains two non-adjacent vertices a, a', and the set $N(w) - N[u]$ contains two non-adjacent vertices b, b'. Now, the graph $G[u, a, a', w, b, b']$ is isomorphic to one of the graphs G_1–G_6, a contradiction. □

Making use of the notation of the proof of Theorem 3, we see that $P_i \neq \emptyset$ for $1 \leqslant i \leqslant l$, for otherwise $D' = (D - \{x_i\}) \cup \{v\}$ would be a minimum dominating set of G with $d(D') > d(D)$. By Claim 1, $d(w) \leqslant 2$ for any $w \in N_0 \cup N_1$. Since $a_i \in N_1$ is dominated by D and $P_i \neq \emptyset$, we obtain $d(w) = 2$ for any $w \in N_0 \cup N_1$, $P_i = \{y_i\}$ for $1 \leqslant i \leqslant l$, and $P_{i,j} = \emptyset$ for $i \neq j$. Suppose that $|N_0| \geqslant 3$. Since $G[v, x_1, y_1, x_2, y_2, x_3, y_3] \ncong T^*$, we may assume w.l.o.g. that $y_1 \perp y_2$. Now $D' = (D - \{x_1, x_2\}) \cup \{v, y_1\}$ is a minimum dominating set and $d(D') > d(D)$, a contradiction. Consider the case $|N_0| = 2$, thus $N_1 \neq \emptyset$ and there are adjacent vertices $a_1 \in N_1$ and $b_1 \in N_2$. By the definition of P_i, $b_1 \pm \{y_1, y_2\}$. Since $G[v, x_1, y_1, x_2, y_2, a_1, b_1] \ncong T^*$, we obtain $y_1 \perp y_2$ and $D' = (D - \{x_1, x_2\}) \cup \{v, y_1\}$ is a dominating set with $d(D') > d(D)$, a contradiction. At last, let $|N_0| = 1$. Since $d(v) \geqslant 3$, there are $a_1, a_2 \in N_1$ and $b_1, b_2 \in N_2$ such that $a_1 \perp b_1$ and $a_2 \perp b_2$. If $b_1 = b_2$, then $G[y_1, x_1, v, a_1, b_1, a_2] \cong \mathrm{PD}$. Hence $b_1 \neq b_2$. Since $G[v, x_1, y_1, a_1, b_1, a_2, b_2] \ncong T^*$, we obtain $b_1 \perp b_2$. By Claim 1, one of the vertices b_1 and b_2, say b_1, has degree 2. Now $D' = (D - \{b_1\}) \cup \{v\}$ is a minimum dominating set and $d(D') > d(D)$, a contradiction. □

Corollary 7. *A bipartite graph G is γi_S -perfect if and only if it does not contain any of the graphs $G_1 - G_6$ in Fig.* 1 *and* PD, T^* *in Fig.* 2 *as an induced subgraph.*

4. ii_S-perfect graphs

The next theorem gives a characterization of $\{C_3, C_5\}$-free ii_S-perfect graphs. Let T' denote a tree obtained from three copies of $K_{1,2}$ by adding two edges connecting their centers.

Theorem 5. *A $\{C_3, C_5\}$-free graph G is ii_S-perfect if and only if it does not contain any of the graphs G_3, G_5 (Fig.* 1), PD, T^* *(Fig.* 2) *and T' as an induced subgraph.*

Proof. The 'only if'-part follows from the fact that $i < i_S$ for any of the forbidden graphs. To prove the 'if'-part, let G be a minimum counterexample, i.e. G is a graph of minimal order such that $i(G) < i_S(G)$, G is $\{C_3, C_5\}$-free and it does not contain any of the above graphs as induced subgraphs. Denote by d^* the maximum of $d(I) = \sum_{x \in I} d(x)$ taken over all minimum independent dominating sets I. From among all minimum independent dominating sets I having $d(I) = d^*$, we choose a set I which strongly dominates as many vertices as possible. Since $i(G) < i_S(G)$, there is a vertex $v \in V(G) - I$ which is not strongly dominated by I. Denote $X = N(v) \cap I$. Note that $X \neq \emptyset$ and $d(v) > d(x)$ for every $x \in X$. If $d(v) = 2$ and $|X| = 2$, then $G \cong P_3$, and if

$d(v) = 2$ and $X = \{x\}$, then the independent set $I' = (I - \{x\}) \cup \{v\}$ has $d(I') > d(I)$, a contradiction. Hence $d(v) \geqslant 3$.

Claim 1. *If* $v \perp \{a, x\}$ *and* $p \perp x$ *where* $x \in I$ *and* $a, v, p \in V(G) - I$, *then* $a \pm p$.

Proof. Suppose to the contrary that $a \perp p$ and denote $B = N(a) \cap I$. Since I is an independent dominating set, there is $b \in B$ and $b \pm x$. Assume that there exists a vertex $w \in N(b) - \{a\}$ and consider the graph $F = G[w, b, a, v, x, p]$. Since G is $\{C_3, C_5\}$-free, the only edges undetermined in F are wv and wp. Therefore, the graph F is isomorphic to PD, G_3 or G_5, a contradiction. Hence $d(b) = 1$ for any $b \in B$. Now, the set $I' = (I - B) \cup \{a\}$ is an independent dominating set and either $|I'| < |I|$ or $|I'| = |I|$ and $d(I') > d(I)$, a contradiction. $\square$

Denote $N^* = N(X) - \{v\}$ and $\mathrm{PN}^* = N^* - N(I - X)$. Suppose that $\mathrm{PN}^* = \emptyset$ and consider the set $I' = (I - X) \cup \{v\}$. We have $|I'| < |I|$ if $|X| \geqslant 2$, and $d(I') > d(I)$ if $|X| = 1$, a contradiction. Hence

$$\emptyset \neq \mathrm{PN}^* \subseteq N^*.$$

Assume that $v \perp \{a, a'\} \subset V(G) - I$, and let $p \in \mathrm{PN}^*$, thus there is an $x \in X$ adjacent to p and v. By Claim 1, $p \pm \{a, a'\}$. Since I is a dominating set, $\{a, a'\}$ is dominated by $I - X$. If there is a $b \in I$ such that $b \perp \{a, a'\}$, then $G[v, a, b, a', x, p] \cong \mathrm{PD}$, otherwise there are $b, b' \in I$ such that $b \perp a$, $b' \perp a'$, $b \pm a'$, $b' \pm a$ and $G[v, a, b, a', b', x, p] \cong T^*$, a contradiction. Hence v is adjacent to at most one vertex of $V(G) - I$.

Now we put $|X| = m$ and suppose that $v \perp a \in V(G) - I$, thus $d(v) = m + 1$. Since $d(v) \geqslant 3$, we have $|X| \geqslant 2$. Let $b \in N(a) \cap I$. Assume that there is a $p \in N^*$ with $p \perp \{x_1, x_2\} \subseteq X$. By Claim 1, $p \pm a$ and also $p \pm b$, for otherwise we would have C_5. Now $G[v, x_1, p, x_2, a, b] \cong \mathrm{PD}$. If there are $p_1, p_2 \in N^*$ such that $p_i \perp x_i \in X$, $p_i \pm x_j$ for $i \neq j \in \{1, 2\}$, then $a \pm \{p_1, p_2\}$ by Claim 1. Also, $p_1 \pm p_2$ and $b \pm \{p_1, p_2\}$, for otherwise we have C_5. Now $G[v, a, b, x_1, p_1, x_2, p_2] \cong T^*$. Thus, $N(p) \cap X = \{x_1\}$ for any $p \in N^*$, and hence $d(x) = 1$ for each $x \in X - \{x_1\}$. We obtain

$$|\mathrm{PN}^*| \leqslant |N^*| = d(x_1) - 1 \leqslant d(v) - 2 = m - 1.$$

Consider the set $I' = (I - X) \cup \{v\} \cup \mathrm{PN}^*$. Since I is a minimum independent dominating set, we get $|I'| = |I|$ and hence $|\mathrm{PN}^*| = m - 1$. We have $d(X) \leqslant 2m - 1$ and $d(\{v\} \cup \mathrm{PN}^*) \geqslant 2m$. Therefore $d(I') > d(I)$, a contradiction.

Let $N(v) = X = \{x_1, \ldots, x_m\}$, $m \geqslant 3$. Suppose that $|N(p) \cap X| = 1$ for every $p \in N^*$, thus $N^* = \bigcup_{i=1}^{m} N_i$ where $N_i = \{p \in N^* \mid N(p) \cap X = \{x_i\}\}$. Since T^* is forbidden, it is easy to see that $N_i = \emptyset$ for $i \geqslant 3$. Hence $d(x_i) = 1$ for $i \geqslant 3$. Put $\mathrm{PN}_i = N_i \cap \mathrm{PN}^*$. Since $\mathrm{PN}^* \neq \emptyset$, we may assume w.l.o.g. that $\mathrm{PN}_1 \neq \emptyset$. Further, $d(v) > d(x_i)$ implies

$$|\mathrm{PN}_i| \leqslant |N_i| \leqslant d(x_i) - 1 \leqslant d(v) - 2 = m - 2.$$

We have $\mathrm{PN}_2 \neq \emptyset$, for otherwise $I' = (I - X) \cup \{v\} \cup \mathrm{PN}_1$ is an independent dominating set and $|I'| < |I|$. Hence $d(x_1) = d(x_2) = 2$ if $m = 3$. If $m \geqslant 4$ and $|N_1| \geqslant 2$, $|N_2| \geqslant 2$, then, taking the set $\{v, x_1, x_2, x_3, x_4\}$, two vertices from N_1 and two vertices from N_2, we obtain T'. Therefore, we may assume w.l.o.g. that $\mathrm{PN}_2 = N_2 = \{p\}$, i.e. $d(x_2) = 2$. Consider the set $I' = (I - X) \cup \mathrm{PN}_1 \cup \{v, p\}$ which is independent dominating. If $|\mathrm{PN}_1| \leqslant m-3$, then $|I'| < |I|$. Hence $|\mathrm{PN}_1| = |N_1| = m-2$ and $d(x_1) = m-1$. If the set $\mathrm{PN}_1 \cup \{p\}$ contains a vertex of degree at least two, then $d(\mathrm{PN}_1 \cup \{v, p\}) > 2m-1$, $d(X) = 2m-1$ and $d(I') > d(I)$. Thus, the set $\mathrm{PN}_1 \cup \{p\}$ consists of pendant vertices only and hence I' strongly dominates v and also vertices strongly dominated by I, a contradiction.

At last, let the set $W = N^* - \bigcup_{i=1}^{m} N_i$ be not empty. Denote $N_X(w) = N(w) \cap X$, thus $|N_X(w)| \geqslant 2$ for any $w \in W$. Suppose that there are vertices $a, b \in W$ such that $N_X(a) \neq N_X(b)$. If $N_X(a) \cap N_X(b) = \emptyset$, then $G[a, x, x', v, x'', b] \cong \mathrm{PD}$ where $x, x' \in N_X(a)$ and $x'' \in N_X(b)$. Now let there be an $x \perp \{a, b\}$ where $x \in X$. Since $N_X(a) \neq N_X(b)$, there exists an $x' \in N_X(a) - N_X(b)$, and let $x'' \in N_X(b) - \{x\}$. We obtain $G[a, b, x, x', x'', v] \cong G_3$ or G_5, a contradiction. Thus $N_X(a) = N_X(b)$ for any $a, b \in W$. We put $N(W) \cap X = \{x_1, \ldots, x_k\}$, $k \geqslant 2$. Further, $d(x_i) = 1$ for $i > k$, for otherwise there is $p \in N_i$ and $G[p, x_i, v, x_1, x_2, w] \cong \mathrm{PD}$ where $w \in W$. Since T^* is forbidden, we may assume w.l.o.g. that $N_i = \emptyset$ for $3 \leqslant i \leqslant k$. If $p \in N_1$ and $p \perp y \neq x_1$, then $y \notin X \cup N^*$, $y \pm \{v, x_1, x_2\}$ and hence $G[y, p, x_1, v, x_2, w] \cong \mathrm{PD}$ or G_3. Thus, $d(p) = 1$ for any $p \in N_1$ and analogously for any $p \in N_2$. This implies $\mathrm{PN}_1 = N_1$ and $\mathrm{PN}_2 = N_2$. Denote $\mathrm{PW} = W \cap \mathrm{PN}^*$. We have for $i = 1, 2$:

$$|N_i| + |PW| \leqslant |N_i| + |W| = d(x_i) - 1 \leqslant d(v) - 2 = m - 2.$$

If $N_2 = \emptyset$, then $I' = (I - X) \cup \{v\} \cup N_1 \cup \mathrm{PW}$ is an independent dominating set and $|I'| < |I|$. Hence $N_2 \neq \emptyset$, and analogously $N_1 \neq \emptyset$. We have $m = d(v) > d(x_1) \geqslant 3$. Now, if $|N_1| > 1$ and $|N_2| > 1$, then $\{x_1, x_2, x_3, x_4, v\}$ with $a, b \in N_1$ and $c, d \in N_2$ induce T'. Hence w.l.o.g. $N_2 = \{p\}$. Consider the set $I' = (I - X) \cup N_1 \cup \mathrm{PW} \cup \{v, p\}$. If $|N_1| + |\mathrm{PW}| < m - 2$, then $|I'| < |I|$, a contradiction. Therefore, $|N_1| + |\mathrm{PW}| = m - 2$ and hence $\mathrm{PW} = W$. Thus, the set I' strongly dominates v and also all vertices strongly dominated by the set I, a contradiction. □

Corollary 8. *A bipartite graph is* ii_S*-perfect if and only if it does not contain any of the graphs* $G_3, G_5, \mathrm{PD}, T^*$ *or* T' *as an induced subgraph.*

Corollary 9. *A graph* G *with girth* $g(G) > 5$ *is* ii_S*-perfect if and only if it does not contain* T^* *and* T' *as induced subgraphs.*

5. Complexity results

We say that a graph G *belongs to the class* $\mathscr{L}$ if G is a bipartite planar graph of maximum degree 3 and girth $g(G) \geqslant k$ for a fixed k. In our next theorem, the strong dominating set (SDS) and independent strong dominating set (ISDS) problems are shown to be both NP-complete on the class $\mathscr{L}$. It is known [11] that the dominating set problem is NP-complete for 3-regular planar graphs. Since the domination number equals the strong domination number for a regular graph, the SDS problem is NP-complete for 3-regular planar graphs. It seems that no result on NP-completeness of the ISDS problem is known. (Note that when we speak of the SDS *problem* or similar problems, we always understand the problem of deciding whether for a given graph G and a given integer k, there is a strong dominating set of G of cardinality at most k.)

Theorem 6. *The* SDS *and* ISDS *problems are both* NP-*complete on the class* $\mathscr{L}$.

Proof. Let us define the operation of Δ-partition of an edge uv whose endvertices have degree 3. We replace uv by the path $P_5=(u,x,y,z,v)$, and also attach the three paths (x,x',x''), (y,y',y'') and (z,z',z''). We prove that if H is obtained from G by Δ-partition of an edge uv of degree 6 (note that the degree of an edge uv equals $d(u)+d(v)$), then $\gamma_S(H)=\gamma_S(G)+4$. Let D be a minimum strong dominating set of G and let $A=\{x',y',z'\}$. If $u,v\in D$ or $u,v\notin D$, then $D\cup A\cup\{y\}$ strongly dominates H. If $u\in D$ and $v\notin D$, then $D\cup A\cup\{z\}$ strongly dominates H, and analogously for the case $u\notin D$ and $v\in D$. Hence $\gamma_S(H)\leqslant\gamma_S(G)+4$. It is easy to see that there is a minimum strong dominating set D of H such that $A\subseteq D$. The set $S=D\cap\{u,x,y,z\}$ is not empty. Suppose that $|S|=1$, in this case either $S=\{x\}$ or $S=\{y\}$. If $S=\{x\}$, then $v\in D$ and the set $D-(\{x\}\cup A)$ strongly dominates G. If $S=\{y\}$, then u,v are strongly dominated by $D-(S\cup A)$ and hence the set $D-(\{x\}\cup A)$ strongly dominates G. If $|S|\geqslant 2$, then the set $(D-(S\cup A))\cup\{u\}$ strongly dominates G. For any case, $\gamma_S(G)\leqslant|D|-4=\gamma_S(H)-4$, i.e., $\gamma_S(H)\geqslant\gamma_S(G)+4$.

Now we describe a polynomial time reduction from the SDS problem for the class of 3-regular planar graphs to the SDS and ISDS problems for some subclass of the class $\mathscr{L}$. Let G be a 3-regular planar graph. Starting with G and repeatedly applying the operation of Δ-partition to edges of degree 6, we can obtain a graph G' with girth at least k. The single Δ-partition of each edge of degree 6 of the graph G' results in a bipartite graph H. Thus, the graph H belongs to $\mathscr{L}$ and H is constructible in polynomial time, since k is fixed. We have $\gamma_S(H)=\gamma_S(G)+4t$, where t is the number of Δ-partitions of G.

It remains to show that $\gamma_S(H)=i_S(H)$. Let D be a minimum strong dominating set of H containing as few edges as possible, and suppose that D has an edge ab. If $d(a)=2$ and $d(b)=1$, then D is not minimum. If $d(a)=3$ and $d(b)=2$, then $b\perp c$ with $d(c)=1$ and $(D-\{b\})\cup\{c\}$ is a minimum strong dominating set containing fewer edges than D, a contradiction. Let $d(a)=d(b)=3$. By the construction, $a\notin V(G)$ or $b\notin V(G)$. W.l.o.g., let $b\notin V(G)$, i.e., $b\perp b'$, $b'\perp b''$ and $d(b')=2$, $d(b'')=1$. Also, $b\perp c$ and $d(c)=3$. Let $PN_S(b)$ be the set of vertices which are not strongly dominated by $D-\{b\}$. Since D is minimum, $PN_S(b)\neq\emptyset$. If $PN_S(b)=\{b'\}$, then $b''\in D$ and $(D-\{b,b''\})\cup\{b'\}$ is a strong dominating set. Suppose that $PN_S(b)=\{c\}$. If $c\in V(G)$, then $(D-\{b\})\cup\{c\}$ is a strong dominating set containing fewer edges than D, a contradiction. If $c\notin V(G)$, then there are c',c'' of degree 2 and 1 such that $c'\in D$ and $c''\notin D$. Now the set $(D-\{b,c'\})\cup\{c,c''\}$ yields a contradiction. At last, let $PN_S(b) = \{b', c\}$. If $c \in V(G)$, then $(D - \{b, b''\}) \cup \{c, b'\}$ gives a contradiction, while if $c \notin D$, then $c' \in D$, $c'' \notin D$ and the set $(D - \{b, b'', c'\}) \cup \{b', c, c''\}$ yields a contradiction. □

Using Theorem 6 we can show that the weak dominating set (WDS) and independent weak dominating set (IWDS) problems are both NP-complete on the class $\mathscr{L}$. The only known result on this subject is due to Hattingh and Laskar [7] who proved that the WDS problem is NP-complete for bipartite graphs. Recall that a set X is a *weak dominating set* if every vertex $u \in V(G) - X$ is adjacent to a vertex $v \in X$ with $d(v) \leqslant d(u)$. The *weak domination number* $\gamma_W(G)$ is the minimum cardinality of a weak dominating set of G, and the *independent weak domination number* $i_W(G)$ is the minimum cardinality of an independent weak dominating set of G. Other results on these parameters can be found in [7–9,15,16].

Theorem 7. *The* WDS *and* IWDS *problems are both* NP-*complete on the class* $\mathscr{L}$.

Proof. Using the proof of the previous theorem, we only need to prove that $\gamma_W(H) = \gamma_S(H)$ and $i_W(H) = i_S(H)$ for the graph H. Let D be a minimum strong dominating set of H. Denote by Z the set of vertices from $V(H) - D$ which are not weakly dominated by D. Thus, if $z \in Z$, then $d(x) > d(z)$ for any $x \in N(z) \cap D$. Since $d(u) \leqslant 3$ for any $u \in V(H)$, we have $d(z) \leqslant 2$. Suppose that $d(z) = 2$. We know that $z \perp p$ with $d(p) = 1$ and hence $p \in D$, since D is a dominating set. Thus $d(p) < d(z)$ and therefore $d(z) = 1$. We have $z \perp u \in D$ and $d(u) = 2$. Consider the vertex w adjacent to u. We obtain $3 = d(w) > d(u)$, and hence either $w \in D$ or $w \notin D$ and $w \perp w' \in D$ with $d(w') = 3$. Thus, the set Z consists of pendant vertices only and $(D - N(Z)) \cup Z$ is a weak dominating set of H. Hence $\gamma_W(H) \leqslant \gamma_S(H)$. Now, let D be a minimum weak dominating set of H. Clearly that $p \in D$ for any vertex p of degree 1. For each $P_3 = (x, x', x'')$ with $d(x) = 3$, $d(x') = 2$, $d(x'') = 1$ we make the following. If $x' \notin D$, then we delete x'' from D and add x' in D. If $x' \in D$, then $x \notin D$, since D is minimum. We delete x' from D and add x in D. The resulting set D' is obviously a strong dominating set of H, thus $\gamma_S(H) \leqslant \gamma_W(H)$. The equality $i_W(H) = i_S(H)$ is proved in the same way. □

6. Conclusion and open problems

We have proved a number of characterization results on strong domination perfect graphs, $\gamma\gamma_S$-perfect graphs, ii_S-perfect graphs, γi_S-perfect graphs as well as results on NP-completeness of the strong dominating set, independent strong dominating set, weak dominating set and independent weak dominating set problems. There are, however, many questions left open. We strongly believe that each of the above classes admits a finite forbidden induced subgraph characterization. Also, it would be of great interest to provide further results towards characterizations of the above classes of '$\pi\pi'$-perfect' graphs. In particular, it is interesting to characterize k-$\pi\pi'$-perfect graphs for small values of k (a graph G is called *k-$\pi\pi'$-perfect* if $\pi(H)=\pi'(H)$, for every induced subgraph H of G with $\pi(H)\leqslant k$). A further problem worth investigating concerns the existence of polynomial time algorithms for computing the value of the parameter $\pi(G)$ for $\pi\pi'$-perfect graphs. It follows from the results of [19] that the dominating set problem is NP-complete even for triangle-free domination perfect graphs. In contrast, it is not difficult to show that the strong domination number can be computed in polynomial time for triangle-free strong domination perfect graphs as well as the domination number can be computed in polynomial time for triangle-free γi_{S}-perfect graphs. The status of the problem on the entire class of $\pi\pi'$-perfect graphs is, however, unknown.

Finally, one may define a graph G to be *weak domination perfect* if $\gamma_{\mathrm{W}}(H)=i_{\mathrm{W}}(H)$, for every induced subgraph H of G, and consider the above problems for weak domination perfect graphs. It was proved in [16] that any $K_{1,3}$-graph is weak domination perfect. We can easily show that a graph G with girth $g(G)>7$ is weak domination perfect if and only if it does not contain T as an induced subgraph, where T is a tree of order 12, and we believe that bipartite weak domination perfect graphs can be characterized in terms of a finite number of forbidden-induced subgraphs.

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