The Binding Number of a Random Graph

Vadim E. Zverovich

Department II of Mathematics
RWTH Aachen, Aachen 52056, Germany

Abstract

Let $G$ be a random graph with $n$ labelled vertices in which the edges are chosen independently with a fixed probability $p$, $0 < p < 1$. In this note we prove that, with the probability tending to 1 as $n \to \infty$, the binding number of a random graph $G$ satisfies:

(i) $b(G) = (n - 1)/(n - \delta)$, where $\delta$ is the minimal degree of $G$;
(ii) $1/q - \epsilon < b(G) < 1/q$, where $\epsilon$ is any fixed positive number and $q = 1 - p$;
(iii) $b(G)$ is realized on a unique set $X = V(G) \setminus N(x)$, where $\deg(x) = \delta(G)$, and the induced subgraph $\langle X \rangle$ contains exactly one isolated vertex $x$.

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All graphs will be finite and undirected, without loops or multiple edges. If $G$ is a graph, $V(G)$ denotes the set of vertices in $G$, and $n = |V(G)|$. We shall denote the neighborhood of a vertex $x$ by $N(x)$. More generally, $N(X) = \bigcup_{x \in X} N(x)$ for $X \subseteq V(G)$. The minimal degree of vertices and the vertex connectivity of $G$ are denoted by $\delta = \delta(G)$ and $\kappa(G)$, respectively. For a set $X$ of vertices, $\langle X \rangle$ denotes the subgraph of $G$ induced by $X$.

Woodall [5] defined the binding number $b(G)$ of a graph $G$ as follows:

$$b(G) = \min_{X \in \mathcal{F}} \frac{|N(X)|}{|X|},$$

where $\mathcal{F} = \{X : \emptyset \neq X \subseteq V(G), N(X) \neq V(G)\}$. We say that $b(G)$ is realized on a set $X$ if $X \in \mathcal{F}$ and $b(G) = |N(X)| / |X|$, and the set $X$ is called a realizing set for $b(G)$.

Proposition 1 For any graph $G$,

$$\frac{\delta}{n - \delta} \leq b(G) \leq \frac{n - 1}{n - \delta}.$$
Proof. The upper bound is proved by Woodall in [5]. Let us prove the lower bound. Let \( X \in \mathcal{F} \) and \( |N(X)| / |X| = b(G) \), i.e., \( X \) is a realizing set. We have \( |N(X)| \geq \delta \), since the set \( X \) is not empty. Suppose that \( |X| \geq n - \delta + 1 \). Then any vertex of \( G \) is adjacent to some vertex of \( X \), i.e. \( N(X) = V(G) \), a contradiction. Therefore \( |X| \leq n - \delta \) and \( b(G) = |N(X)| / |X| \geq \delta/(n-\delta) \). The proof is complete.

Note that the difference between the upper and lower bounds on \( b(G) \) in Proposition 1 is less than 1. In the sequel we shall see that the binding number of almost every graph is equal to the upper bound in Proposition 1.

Let \( 0 < p < 1 \) be fixed and put \( q = 1 - p \). Denote by \( \mathcal{G}(n, p) \) the discrete probability space consisting of all graphs with \( n \) fixed and labelled vertices, in which the probability of each graph with \( M \) edges is \( p^M q^{N-M} \), where \( N = \binom{n}{2} \).

Equivalently, the edges of a labelled random graph are chosen independently and with the same probability \( p \). We say that a random graph \( G \) satisfies a property \( Q \) if

\[
P(G \text{ has } Q) \to 1 \text{ as } n \to \infty.
\]

We shall need the following results.

Theorem 1 (Bollobás [1]) A random graph \( G \) satisfies \( \kappa(G) = \delta(G) \).

Theorem 2 (Bollobás [1]) A random graph \( G \) satisfies

\[
|\delta(G) - pn + (2pnq \log n)^{1/2} - \left(\frac{pqn}{8 \log n}\right)^{1/2} \log \log n| \leq C(n) \left(\frac{n}{\log n}\right)^{1/2},
\]

where \( C(n) \to \infty \) arbitrarily slowly.

Theorem 3 (Erdős and Wilson [3]) A random graph has a unique vertex of minimal degree.

Now we can state the main result of the paper.

Theorem 4 The binding number of a random graph \( G \) satisfies

\[
b(G) = \frac{n - 1}{n - \delta}.
\]

Proof. Taking into account Proposition 1, it is sufficient to prove that

\[
\frac{|N(X)|}{|X|} \geq \frac{n - 1}{n - \delta}
\]

for any set \( X \in \mathcal{F} \). Let \( Y = N(X) \setminus X \) and consider three cases.

(i) The induced subgraph \( \langle X \rangle \) does not contain an isolated vertex. The set \( V(G) \setminus N(X) \) is not empty, since \( X \in \mathcal{F} \). Hence the set \( Y \) is a cutset of the graph \( G \). By Theorem 1, \( \kappa(G) = \delta(G) \). Therefore \( |Y| \geq \delta \) and \( |X| < n - \delta \). We have

\[
\frac{|N(X)|}{|X|} = \frac{|Y| + |X|}{|X|} = \frac{|Y|}{|X|} + 1 \geq \frac{n}{n - \delta} > \frac{n - 1}{n - \delta}.
\]
(ii) The induced subgraph $\langle X \rangle$ contains exactly one isolated vertex. Obviously $|Y| \geq \delta$ and $|X| \leq n - \delta$. Then, taking into account that $\delta(G) > 0$, we obtain

$$\frac{|N(X)|}{|X|} = \frac{|Y| + |X| - 1}{|X|} = \frac{|Y| - 1}{|X|} + 1 \geq \frac{n - 1}{n - \delta}.$$ 

(iii) The induced subgraph $\langle X \rangle$ contains more than one isolated vertex. If $x$ and $y$ are different vertices of $G$, then $\deg(x, y)$ denotes the pair degree of the vertices $x$ and $y$, i.e., the cardinality $|N\{x, y\}| \backslash \{x, y\}$. Define $\mu = \mu(G) = \min \deg(x, y)$, where the minimum is taken over all pairs of different vertices $x, y \in V(G)$. Now introduce a random variable $\xi$ on $\mathcal{G}(n, \mathbb{P}(\text{edge}) = p)$. The random variable $\xi$ is equal to the number of pairs of different vertices in $G$ such that $\deg(x, y) \leq (1 - q^2 - \epsilon)(n - 2)$, where $\epsilon$ is fixed and $0 < \epsilon < 1 - q^2$. We need to estimate the expectation $\mathbb{E}\xi$. Let the vertices $x$ and $y$ be fixed. Then

$$\Pi = \mathbb{P}(\deg(x, y) \leq k) = \sum_{t \leq k} \binom{m}{t} (1 - q^2)^t (q^2)^{n-2-t},$$

where $k = (n - 2)(1 - q^2 - \epsilon)$. We now use the Chernoff formula [2]:

$$\sum_{t \leq k} \binom{m}{t} P^t Q^{m-t} \leq \exp \left( k \log \frac{mP}{k} + (m - k) \log \frac{mQ}{m-k} \right)$$

whenever $k \leq mP, P > 0, Q > 0$ and $P + Q = 1$. Taking $m = n - 2, k = m(1 - q^2 - \epsilon), P = 1 - q^2$ and $Q = q^2$, and noting that $\log x < x - 1$ if $x \neq 1$, we find that

$$\Pi \leq \exp\{(n - 2)\Theta\}$$

where

$$\Theta = (1 - q^2 - \epsilon) \log \frac{1 - q^2}{1 - q^2 - \epsilon} + (q^2 + \epsilon) \log \frac{q^2}{q^2 + \epsilon}$$

$$< (1 - q^2) - (1 - q^2 - \epsilon) + q^2 - (q^2 + \epsilon) = 0.$$ 

Thus $\Pi < e^{-Cn}$, where $C > 0$ is a constant. At last, we get

$$\mathbb{E}\xi \leq \binom{n}{2} e^{-Cn} = o(1).$$

If $\xi$ is a non-negative random variable with expectation $\mathbb{E}\xi > 0$ and $r > 0$, then from the Markov inequality it follows that

$$\mathbb{P}(\xi \geq r \mathbb{E}\xi) \leq 1/r.$$ 

Taking $r = 1/\mathbb{E}\xi$, we have $\mathbb{P}(\xi \geq 1) \leq \mathbb{E}\xi = o(1)$, i.e. $\mathbb{P}(\xi = 0) = 1 - o(1)$. Thus

$$\mu > (1 - q^2 - \epsilon)(n - 2).$$
Denote by \( m \) the number of isolated vertices in the graph \( \langle X \rangle \). Clearly \( m \leq \alpha \), where \( \alpha = \alpha(G) \) is the independence number of \( G \). It is well-known [4] that for a random graph \( G, \alpha(G) = o(n) \), so that \( \mu > \alpha \). Furthermore, \( |Y| \geq \mu \) and \( |X| \leq n - \mu \), since \( m \geq 2 \), and so \( |Y| - m \geq \mu - \alpha > 0 \). We obtain

\[
\frac{|N(X)|}{|X|} = \frac{|Y| + |X| - m}{|X|} = \frac{|Y| - m}{|X|} + 1 \geq \frac{\mu - \alpha}{n - \mu} + 1 = \frac{n - \alpha}{n - \mu} > \frac{n - o(n)}{n - (1 - q^2 - \epsilon)(n - 2)} = \frac{1}{\epsilon + q^2(1 - o(1))}.
\]

On the other hand, by Theorem 2,

\[
\frac{n - 1}{n - \delta} = \frac{n - 1}{n - pn(1 - o(1))} = \frac{1}{q}(1 - o(1)).
\]

Now, if we take \( \epsilon < q - q^2 \), then we have

\[
\frac{|N(X)|}{|X|} > \frac{n - 1}{n - \delta}.
\]

This completes the proof of Theorem 4. ■

Using Theorems 2-4, the following corollaries are obtained.

**Corollary 1** If \( C(n) \to \infty \) arbitrarily slowly, then the binding number of a random graph \( G \) satisfies

\[
\frac{n - 1}{K + C(n)(n/ \log n)^{1/2}} \leq b(G) \leq \frac{n - 1}{K - C(n)(n/ \log n)^{1/2}},
\]

where

\[
K = qn + (2pq n \log n)^{1/2} - \left(\frac{pq n}{8 \log n}\right)^{1/2} \log \log n.
\]

The proof follows immediately from Theorems 2 and 4. ■

It may be pointed out that the bounds in Corollary 1 are essentially best possible, since the result of Theorem 2 is best possible (see [1]).

**Corollary 2** If \( \epsilon > 0 \) is fixed, then the binding number of a random graph \( G \) satisfies

\[
1/q - \epsilon < b(G) < 1/q.
\]

The proof follows immediately from Corollary 1. ■

**Corollary 3** The binding number of a random graph \( G \) is realized on a unique set \( X = V(G) \setminus N(x) \), where \( \deg(x) = \delta(G) \), and the graph \( \langle X \rangle \) contains exactly one isolated vertex \( x \).

**Proof.** One may see from the proof of Theorem 4 that the equality

\[
\frac{|N(X)|}{|X|} = (n - 1)/(n - \delta)
\]

for a random graph \( G \) is possible only if the graph \( \langle X \rangle \) contains exactly one isolated vertex \( x \) and \( |X| = n - \delta \). Thus \( \deg(x) = \delta(G) \) and \( X = V(G) \setminus N(x) \). By Theorem 3, the set \( X \) is unique. ■

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References


