The Binding Number of a Random Graph

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Abstract

Let $\mathbf{G}$ be a random graph with n labelled vertices in which the edges are chosen independently with a fixed probability p, $0 < p < 1$. In this note we prove that, with the probability tending to 1 as $n \to \infty$, the binding number of a random graph $\mathbf{G}$ satisfies:

(i) $b(\mathbf{G}) = (n-1)/(n-\delta)$, where δ is the minimal degree of $\mathbf{G}$;

(ii) $1/q - \epsilon < b(\mathbf{G}) < 1/q$, where ϵ is any fixed positive number and $q = 1-p$;

(iii) $b(\mathbf{G})$ is realized on a unique set $X = V(\mathbf{G}) \backslash N(x)$, where $\deg(x) = \delta(\mathbf{G})$, and the induced subgraph $\langle X \rangle$ contains exactly one isolated vertex x.

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All graphs will be finite and undirected, without loops or multiple edges. If G is a graph, $V(G)$ denotes the set of vertices in G, and $n = \mid V(G) \mid$. We shall denote the neighborhood of a vertex x by $N(x)$. More generally, $N(X) = \bigcup_{x \in X} N(x)$ for $X \subseteq V(G)$. The minimal degree of vertices and the vertex connectivity of G are denoted by $\delta = \delta(G)$ and $\kappa(G)$, respectively. For a set X of vertices, $\langle X \rangle$ denotes the subgraph of G induced by X.

Woodall [5] defined the *binding number* $b(G)$ of a graph G as follows:

$$b(G) = \min_{X \in \mathcal{F}} \frac{\mid N(X) \mid}{\mid X \mid},$$

where $\mathcal{F} = \{X : \emptyset \neq X \subseteq V(G), N(X) \neq V(G)\}$. We say that $b(G)$ is *realized* on a set X if $X \in \mathcal{F}$ and $b(G) = \mid N(X) \mid / \mid X \mid$, and the set X is called a *realizing set* for $b(G)$.

Proposition 1 *For any graph G,*

$$\frac{\delta}{n-\delta} \leq b(G) \leq \frac{n-1}{n-\delta}.$$

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Proof. The upper bound is proved by Woodall in [5]. Let us prove the lower bound. Let $X \in \mathcal{F}$ and $\mid N(X) \mid / \mid X \mid = b(G)$, i.e., X is a realizing set. We have $\mid N(X) \mid \geq \delta$, since the set X is not empty. Suppose that $\mid X \mid \geq n - \delta + 1$. Then any vertex of G is adjacent to some vertex of X, i.e. $N(X) = V(G)$, a contradiction. Therefore $\mid X \mid \leq n - \delta$ and $b(G) = \mid N(X) \mid / \mid X \mid \geq \delta/(n-\delta)$. The proof is complete. ∎

Note that the difference between the upper and lower bounds on $b(G)$ in Proposition 1 is less than 1. In the sequel we shall see that the binding number of almost every graph is equal to the upper bound in Proposition 1.

Let $0 < p < 1$ be fixed and put $q = 1 - p$. Denote by $\mathcal{G}(n, \mathbf{P}(edge) = p)$ the discrete probability space consisting of all graphs with n fixed and labelled vertices, in which the probability of each graph with M edges is $p^M q^{N-M}$, where $N = \binom{n}{2}$. Equivalently, the edges of a labelled random graph are chosen independently and with the same probability p. We say that a random graph $\mathbf{G}$ *satisfies a property* Q if

$$\mathbf{P}(\mathbf{G} \text{ has } Q) \to 1 \text{ as } n \to \infty.$$

We shall need the following results.

Theorem 1 (Bollobás [1]) *A random graph* $\mathbf{G}$ *satisfies* $\kappa(\mathbf{G}) = \delta(\mathbf{G})$.

Theorem 2 (Bollobás [1]) *A random graph* $\mathbf{G}$ *satisfies*

$$\mid \delta(\mathbf{G}) - pn + (2pqn\log n)^{1/2} - \Big(\frac{pqn}{8\log n}\Big)^{1/2} \log\log n \mid \leq C(n)\Big(\frac{n}{\log n}\Big)^{1/2},$$

where $C(n) \to \infty$ *arbitrarily slowly.*

Theorem 3 (Erdös and Wilson [3]) *A random graph has a unique vertex of minimal degree.*

Now we can state the main result of the paper.

Theorem 4 *The binding number of a random graph* $\mathbf{G}$ *satisfies*

$$b(\mathbf{G}) = \frac{n-1}{n-\delta}.$$

Proof. Taking into account Proposition 1, it is sufficient to prove that

$$\frac{\mid N(X) \mid}{\mid X \mid} \geq \frac{n-1}{n-\delta}$$

for any set $X \in \mathcal{F}$. Let $Y = N(X) \backslash X$ and consider three cases.

(i) The induced subgraph $\langle X \rangle$ does not contain an isolated vertex. The set $V(\mathbf{G}) \backslash N(X)$ is not empty, since $X \in \mathcal{F}$. Hence the set Y is a cutset of the graph $\mathbf{G}$. By Theorem 1, $\kappa(\mathbf{G}) = \delta(\mathbf{G})$. Therefore $\mid Y \mid \geq \delta$ and $\mid X \mid < n - \delta$. We have

$$\frac{\mid N(X) \mid}{\mid X \mid} = \frac{\mid Y \mid + \mid X \mid}{\mid X \mid} = \frac{\mid Y \mid}{\mid X \mid} + 1 \geq \frac{n}{n-\delta} > \frac{n-1}{n-\delta}.$$

(ii) The induced subgraph $\langle X \rangle$ contains exactly one isolated vertex. Obviously $| Y | \geq \delta$ and $| X | \leq n - \delta$. Then, taking into account that $\delta(\mathbf{G}) > 0$, we obtain

$$\frac{| N(X) |}{| X |} = \frac{| Y | + | X | - 1}{| X |} = \frac{| Y | - 1}{| X |} + 1 \geq \frac{n-1}{n-\delta}.$$

(iii) The induced subgraph $\langle X \rangle$ contains more than one isolated vertex. If x and y are different vertices of $\mathbf{G}$, then $\deg(x, y)$ denotes the *pair degree* of the vertices x and y, i.e., the cardinality $| N(\{x, y\}) \backslash \{x, y\} |$. Define $\mu = \mu(\mathbf{G}) = \min \deg(x, y)$, where the minimum is taken over all pairs of different vertices $x, y \in V(\mathbf{G})$. Now introduce a random variable ξ on $\mathcal{G}(n, \mathbf{P}(edge) = p)$. The random variable ξ is equal to the number of pairs of different vertices in $\mathbf{G}$ such that

$$\deg(x, y) \leq (1 - q^2 - \epsilon)(n - 2),$$

where ϵ is fixed and $0 < \epsilon < 1 - q^2$. We need to estimate the expectation $\mathbf{E}\xi$. Let the vertices x and y be fixed. Then

$$\Pi = \mathbf{P}(\deg(x, y) \leq k) = \sum_{t \leq k} \binom{n-2}{t} (1 - q^2)^t (q^2)^{n-2-t},$$

where $k = (n - 2)(1 - q^2 - \epsilon)$. We now use the Chernoff formula [2]:

$$\sum_{t \leq k} \binom{m}{t} P^t Q^{m-t} \leq \exp\left(k \log \frac{mP}{k} + (m - k) \log \frac{mQ}{m - k}\right)$$

whenever $k \leq mP$, $P > 0$, $Q > 0$ and $P + Q = 1$. Taking $m = n-2$, $k = m(1-q^2-\epsilon)$, $P = 1 - q^2$ and $Q = q^2$, and noting that $\log x < x - 1$ if $x \neq 1$, we find that

$$\Pi \leq \exp\{(n - 2)\Theta\}$$

where

$$\begin{aligned}\Theta &= (1 - q^2 - \epsilon) \log \frac{1 - q^2}{1 - q^2 - \epsilon} + (q^2 + \epsilon) \log \frac{q^2}{q^2 + \epsilon} \\ &< (1 - q^2) - (1 - q^2 - \epsilon) + q^2 - (q^2 + \epsilon) = 0.\end{aligned}$$

Thus $\Pi < e^{-Cn}$, where $C > 0$ is a constant. At last, we get

$$\mathbf{E}\xi \leq \binom{n}{2} e^{-Cn} = o(1).$$

If ξ is a non-negative random variable with expectation $\mathbf{E}\xi > 0$ and $r > 0$, then from the Markov inequality it follows that

$$\mathbf{P}(\xi \geq r\mathbf{E}\xi) \leq 1/r.$$

Taking $r = 1/\mathbf{E}\xi$, we have $\mathbf{P}(\xi \geq 1) \leq \mathbf{E}\xi = o(1)$, i.e. $\mathbf{P}(\xi = 0) = 1 - o(1)$. Thus

$$\mu > (1 - q^2 - \epsilon)(n - 2).$$

Denote by m the number of isolated vertices in the graph $\langle X \rangle$. Clearly $m \leq \alpha$, where $\alpha = \alpha(\mathbf{G})$ is the independence number of $\mathbf{G}$. It is well-known [4] that for a random graph $\mathbf{G}, \alpha(\mathbf{G}) = o(n)$, so that $\mu > \alpha$. Furthermore, $|Y| \geq \mu$ and $|X| \leq n - \mu$, since $m \geq 2$, and so $|Y| - m \geq \mu - \alpha > 0$. We obtain

$$\frac{\mid N(X) \mid}{\mid X \mid} = \frac{\mid Y \mid + \mid X \mid - m}{\mid X \mid} = \frac{\mid Y \mid - m}{\mid X \mid} + 1 \geq \frac{\mu - \alpha}{n - \mu} + 1 =$$
$$\frac{n - \alpha}{n - \mu} > \frac{n - o(n)}{n - (1 - q^2 - \epsilon)(n - 2)} = \frac{1}{\epsilon + q^2}(1 - o(1)).$$

On the other hand, by Theorem 2,

$$\frac{n-1}{n-\delta} = \frac{n-1}{n - pn(1 - o(1))} = \frac{1}{q}(1 - o(1)).$$

Now, if we take $\epsilon < q - q^2$, then we have

$$\frac{\mid N(X) \mid}{\mid X \mid} > \frac{n-1}{n-\delta}.$$

This completes the proof of Theorem 4. ∎

Using Theorems 2-4, the following corollaries are obtained.

Corollary 1 *If $C(n) \to \infty$ arbitrarily slowly, then the binding number of a random graph* $\mathbf{G}$ *satisfies*

$$\frac{n-1}{K + C(n)(n/\log n)^{1/2}} \leq b(\mathbf{G}) \leq \frac{n-1}{K - C(n)(n/\log n)^{1/2}},$$

where

$$K = qn + (2pqn \log n)^{1/2} - \Big(\frac{pqn}{8 \log n}\Big)^{1/2} \log \log n.$$

The proof follows immediately from Theorems 2 and 4. ∎

It may be pointed out that the bounds in Corollary 1 are essentially best possible, since the result of Theorem 2 is best possible (see [1]).

Corollary 2 *If $\epsilon > 0$ is fixed, then the binding number of a random graph* $\mathbf{G}$ *satisfies*

$$1/q - \epsilon < b(\mathbf{G}) < 1/q.$$

The proof follows immediately from Corollary 1. ∎

Corollary 3 *The binding number of a random graph* $\mathbf{G}$ *is realized on a unique set* $X = V(\mathbf{G}) \backslash N(x)$, *where* $\deg(x) = \delta(\mathbf{G})$, *and the graph* $\langle X \rangle$ *contains exactly one isolated vertex* x.

Proof. One may see from the proof of Theorem 4 that the equality

$$\mid N(X) \mid / \mid X \mid = (n-1)/(n-\delta)$$

for a random graph $\mathbf{G}$ is possible only if the graph $\langle X \rangle$ contains exactly one isolated vertex x and $\mid X \mid = n - \delta$. Thus $\deg(x) = \delta(\mathbf{G})$ and $X = V(\mathbf{G}) \backslash N(x)$. By Theorem 3, the set X is unique. ∎

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References

[1] B. Bollobás, Degree sequences of random graphs, *Discrete Math.* **33** (1981) 1-19.

[2] H. Chernoff, A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Ann. Math. Stat.* **23** (1952) 493-509.

[3] P. Erdös and R.J. Wilson, On the chromatic index of almost all graphs, *J. Combinatorial Theory Ser.* **B 23** (1977) 255-257.

[4] K. Weber, Random graphs - a survey, *Rostock. Math. Kolloq.* **21** (1982) 83-98.

[5] D.R. Woodall, The binding number of a graph and its Anderson number, *J. Combinatorial Theory Ser.* **B 15** (1973) 225-255.