

The k -tuple domination number revisited

Vadim Zverovich

Faculty of Computing, Engineering and Mathematical Sciences, University of the West of England, Bristol, BS16 1QY, UK

Received 5 October 2006; received in revised form 26 July 2007; accepted 18 October 2007

Abstract

The following fundamental result for the domination number $\gamma(G)$ of a graph G was proved by Alon and Spencer, Arnaoutov, Lovász and Payan:

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1}n,$$

where n is the order and δ is the minimum degree of vertices of G . A similar upper bound for the double domination number was found by Harant and Henning [J. Harant, M.A. Henning, On double domination in graphs, *Discuss. Math. Graph Theory* 25 (2005) 29–34], and for the triple domination number by Rautenbach and Volkmann [D. Rautenbach, L. Volkmann, New bounds on the k -domination number and the k -tuple domination number, *Appl. Math. Lett.* 20 (2007) 98–102], who also posed the interesting conjecture on the k -tuple domination number: for any graph G with $\delta \geq k - 1$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\delta - k + 2}n,$$

where $\widehat{d}_m = \sum_{i=1}^n \binom{d_i}{m} / n$ is the m -degree of G . This conjecture, if true, would generalize all the mentioned upper bounds and improve an upper bound proved in [A. Gagarin, V. Zverovich, A generalised upper bound for the k -tuple domination number, *Discrete Math.* (2007), in press (doi:10.1016/j.disc.2007.07.033)].

In this paper, we prove the Rautenbach–Volkmann conjecture.

© 2007 Elsevier Ltd. All rights reserved.

Keywords: Graphs; Domination number; Double domination; Triple domination; k -tuple domination

1. Notation

All graphs will be finite and undirected without loops and multiple edges. If G is a graph of order n , then $V(G) = \{v_1, v_2, \dots, v_n\}$ is the set of vertices in G , d_i denotes the degree of v_i and $d = \sum_{i=1}^n d_i / n$ is the average degree of G . Let $N(x)$ denote the neighbourhood of a vertex x . Also let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of vertices of G , respectively. Put $\delta = \delta(G)$ and $\Delta = \Delta(G)$. A set X is called a dominating set if every vertex not in X is adjacent to a vertex in X . The minimum

E-mail address: vadim.zverovich@uwe.ac.uk.

cardinality of a dominating set of G is the domination number $\gamma(G)$. A set X is called a k -tuple dominating set of G if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. The minimum cardinality of a k -tuple dominating set of G is the k -tuple domination number $\gamma_{\times k}(G)$. The k -tuple domination number is only defined for graphs with $\delta \geq k - 1$. It is easy to see that $\gamma(G) = \gamma_{\times 1}(G)$ and $\gamma_{\times k}(G) \leq \gamma_{\times k'}(G)$ for $k \leq k'$. The 2-tuple domination number $\gamma_{\times 2}(G)$ is called the double domination number and the 3-tuple domination number $\gamma_{\times 3}(G)$ is called the triple domination number. A number of interesting results on the k -tuple domination number can be found in [3–8,11].

2. Introduction

The following fundamental result was proved by many authors:

Theorem 1 ([1,2,9,10]). For any graph G ,

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1}n.$$

A similar upper bound for the double domination number was found by Harant and Henning [4]:

Theorem 2 ([4]). For any graph G with $\delta \geq 1$,

$$\gamma_{\times 2}(G) \leq \frac{\ln \delta + \ln(\delta + 1) + 1}{\delta}n.$$

Rautenbach and Volkmann posed the following interesting conjecture for the k -tuple domination number:

Conjecture 1 ([11]). For any graph G with $\delta \geq k - 1$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln\left(\sum_{i=1}^n \binom{d_i + 1}{k - 1}\right) - \ln(n) + 1}{\delta - k + 2}n.$$

For $m \leq \delta$, let us define the m -degree \widehat{d}_m of a graph G as follows:

$$\widehat{d}_m = \widehat{d}_m(G) = \sum_{i=1}^n \binom{d_i}{m} / n.$$

Note that \widehat{d}_1 is the average degree d of a graph and $\widehat{d}_0 = 1$. Also, we put $\widehat{d}_{-1} = 0$.

Since

$$\binom{d_i + 1}{k - 1} = \binom{d_i}{k - 1} + \binom{d_i}{k - 2},$$

we see that the above conjecture can be re-formulated as follows:

Conjecture 1'. For any graph G with $\delta \geq k - 1$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\delta - k + 2}n.$$

It may be pointed out that this conjecture, if true, would generalize [Theorem 2](#) and also [Theorem 1](#) taking into account that $\widehat{d}_{-1} = 0$. Rautenbach and Volkmann proved the above conjecture for the triple domination number:

Theorem 3 ([11]). For any graph G with $\delta \geq 2$,

$$\gamma_{\times 3}(G) \leq \frac{\ln(\delta - 1) + \ln(\widehat{d}_2 + d) + 1}{\delta - 1}n.$$

The next result generalizes all the above theorems, but it is still far from **Conjecture 1'**.

Theorem 4 ([3]). For any graph G with $\delta \geq k - 1$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln\left(\sum_{m=1}^{k-1} (k - m)\widehat{d}_m + \epsilon\right) + 1}{\delta - k + 2}n,$$

where $\epsilon = 1$ if $k = 1$ or 2 , and $\epsilon = -d$ if $k \geq 3$.

3. Proof of the conjecture

The following theorem proves the Rautenbach–Volkman conjecture.

Theorem 5. For any graph G with $\delta \geq k - 1$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\delta - k + 2}n.$$

Proof. Let A be a set formed by an independent choice of vertices of G , where each vertex is selected with the probability p , $0 \leq p \leq 1$. For $m = 0, 1, \dots, k - 1$, let us denote

$$B_m = \{v_i \in V(G) - A : |N(v_i) \cap A| = m\}.$$

Also, for $m = 0, 1, \dots, k - 2$, we denote

$$A_m = \{v_i \in A : |N(v_i) \cap A| = m\}.$$

For each set A_m , we form a set A'_m in the following way. For every vertex in the set A_m , we take $k - m - 1$ neighbours not in A and add them to A'_m . Such neighbours always exist because $\delta \geq k - 1$. It is obvious that $|A'_m| \leq (k - m - 1)|A_m|$. For each set B_m , we form a set B'_m by taking $k - m - 1$ neighbours not in A for every vertex in B_m . We have $|B'_m| \leq (k - m - 1)|B_m|$.

We construct the set D as follows:

$$D = A \cup \left(\bigcup_{m=0}^{k-2} A'_m\right) \cup \left(\bigcup_{m=0}^{k-1} B_m \cup B'_m\right).$$

The set D is a k -tuple dominating set. Indeed, if there is a vertex v which is not k -tuple dominated by D , then v is not k -tuple dominated by A . Therefore, v would belong to A_m or B_m for some m , but all such vertices are k -tuple dominated by the set D by construction.

The expected value of $|D|$ is

$$\begin{aligned} E(|D|) &\leq E\left(|A| + \sum_{m=0}^{k-2} |A'_m| + \sum_{m=0}^{k-1} |B_m| + \sum_{m=0}^{k-1} |B'_m|\right) \\ &\leq E\left(|A| + \sum_{m=0}^{k-2} (k - m - 1)|A_m| + \sum_{m=0}^{k-1} (k - m)|B_m|\right) \\ &= E(|A|) + \sum_{m=0}^{k-2} (k - m - 1)E(|A_m|) + \sum_{m=0}^{k-1} (k - m)E(|B_m|). \end{aligned}$$

We have

$$E(|A|) = \sum_{i=1}^n P(v_i \in A) = pn.$$

Also,

$$\begin{aligned} E(|A_m|) &= \sum_{i=1}^n P(v_i \in A_m) \\ &= \sum_{i=1}^n p \binom{d_i}{m} p^m (1-p)^{d_i-m} \\ &\leq p^{m+1} (1-p)^{\delta-m} \sum_{i=1}^n \binom{d_i}{m} \\ &= p^{m+1} (1-p)^{\delta-m} \widehat{d}_m n \end{aligned}$$

and

$$\begin{aligned} E(|B_m|) &= \sum_{i=1}^n P(v_i \in B_m) \\ &= \sum_{i=1}^n (1-p) \binom{d_i}{m} p^m (1-p)^{d_i-m} \\ &\leq p^m (1-p)^{\delta-m+1} \sum_{i=1}^n \binom{d_i}{m} \\ &= p^m (1-p)^{\delta-m+1} \widehat{d}_m n. \end{aligned}$$

Taking into account that $\widehat{d}_{-1} = 0$, we obtain

$$\begin{aligned} E(|D|) &\leq pn + \sum_{m=0}^{k-2} (k-m-1) p^{m+1} (1-p)^{\delta-m} \widehat{d}_m n + \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{\delta-m+1} \widehat{d}_m n \\ &= pn + \sum_{m=1}^{k-1} (k-m) p^m (1-p)^{\delta-m+1} \widehat{d}_{m-1} n + \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{\delta-m+1} \widehat{d}_m n \\ &= pn + \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{\delta-m+1} (\widehat{d}_{m-1} + \widehat{d}_m) n \\ &= pn + (1-p)^{\delta-k+2} n \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{k-m-1} (\widehat{d}_{m-1} + \widehat{d}_m). \end{aligned}$$

Let us denote

$$\mu = \delta - k + 2.$$

Using the inequality $1 - x \leq e^{-x}$, we obtain

$$(1-p)^{\delta-k+2} = (1-p)^\mu \leq e^{-p\mu}.$$

Thus,

$$E(|D|) \leq pn + e^{-p\mu} n \Theta,$$

where

$$\Theta = \sum_{m=0}^{k-1} (k-m) p^m (1-p)^{k-m-1} (\widehat{d}_m + \widehat{d}_{m-1}). \quad (1)$$

We will prove that

$$\Theta \leq \widehat{d}_{k-1} + \widehat{d}_{k-2}.$$

We have

$$\begin{aligned} \Theta &= \sum_{m=0}^{k-1} (k-m)(\widehat{d}_m + \widehat{d}_{m-1}) \sum_{i=0}^{k-m-1} (-1)^i \binom{k-m-1}{i} p^{m+i} \\ &= k(\widehat{d}_0 + \widehat{d}_{-1}) \binom{k-1}{0} p^0 - k(\widehat{d}_0 + \widehat{d}_{-1}) \binom{k-1}{1} p^1 + \dots + k(\widehat{d}_0 + \widehat{d}_{-1}) \binom{k-1}{k-1} (-1)^{k-1} p^{k-1} \\ &\quad + (k-1)(\widehat{d}_1 + \widehat{d}_0) \binom{k-2}{0} p^1 + \dots + (k-1)(\widehat{d}_1 + \widehat{d}_0) \binom{k-2}{k-2} (-1)^{k-2} p^{k-1} \\ &\quad \dots \\ &\quad + (1)(\widehat{d}_{k-1} + \widehat{d}_{k-2}) \binom{0}{0} (-1)^0 p^{k-1} \\ &= \sum_{j=0}^{k-1} \left(\sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} (i+j+1)(\widehat{d}_{k-i-j-1} + \widehat{d}_{k-i-j-2}) \right) p^{k-j-1} \\ &= \sum_{j=0}^{k-1} s_j p^{k-j-1}, \end{aligned}$$

where

$$\begin{aligned} s_j &= \sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} (i+j+1)(\widehat{d}_{k-i-j-1} + \widehat{d}_{k-i-j-2}) \quad (\text{taking into account that } \widehat{d}_{-1} = 0) \\ &= \sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} (i+j+1)\widehat{d}_{k-i-j-1} + \sum_{i=0}^{k-j-2} (-1)^i \binom{i+j}{i} (i+j+1)\widehat{d}_{k-i-j-2} \\ &= \binom{j}{0} (j+1)\widehat{d}_{k-j-1} + \sum_{i=1}^{k-j-1} (-1)^i \binom{i+j}{i} (i+j+1)\widehat{d}_{k-i-j-1} \\ &\quad + \sum_{i=1}^{k-j-1} (-1)^{i-1} \binom{i+j-1}{i-1} (i+j)\widehat{d}_{k-i-j-1} \\ &= (j+1)\widehat{d}_{k-j-1} + \sum_{i=1}^{k-j-1} (-1)^i (j+1) \binom{i+j}{i} \widehat{d}_{k-i-j-1} \\ &= (j+1) \sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} \widehat{d}_{k-i-j-1} \\ &= (j+1) \sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} \sum_{l=1}^n \binom{d_l}{k-i-j-1} / n \\ &= (j+1) \sum_{l=1}^n \sum_{i=0}^{k-j-1} (-1)^i \binom{i+j}{i} \binom{d_l}{k-i-j-1} / n \\ &= (j+1) \sum_{l=1}^n \binom{d_l - j - 1}{k - j - 1} / n \quad (\text{by Lemma 3}) \\ &\geq 0. \end{aligned}$$

Thus, the function $\Theta(p) = s_0 p^{k-1} + s_1 p^{k-2} + \dots + s_{k-1}$ is monotonously increasing in $0 \leq p \leq 1$. Therefore, (1) implies

$$\Theta \leq \widehat{d}_{k-1} + \widehat{d}_{k-2}.$$

We obtain

$$E(|D|) \leq pn + e^{-p\mu} n \Theta \leq pn + e^{-p\mu} n (\widehat{d}_{k-1} + \widehat{d}_{k-2}).$$

Let us denote

$$f(p) = pn + e^{-p\mu} n (\widehat{d}_{k-1} + \widehat{d}_{k-2}).$$

For $p \in [0, 1]$, the function $f(p)$ is minimised at the point $\min\{1, z\}$, where

$$z = \frac{\ln \mu + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2})}{\mu}.$$

There are two cases to consider.

If $z \leq 1$, then

$$E(|D|) \leq f(z) = \left(z + \frac{1}{\mu}\right) n = \frac{\ln \mu + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\mu} n.$$

Since the expected value is an average value, there exists a particular k -tuple dominating set of order at most $f(z)$, as required.

Suppose now that $z > 1$. Taking into account that $\mu > 0$, we obtain

$$\gamma_{\times k}(G) \leq n < \left(z + \frac{1}{\mu}\right) n = \frac{\ln \mu + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\mu} n,$$

as required. The proof of [Theorem 5](#) is complete. ■

For $s \geq 1$, let us denote

$$T_t^s = \binom{s}{t} - \binom{s}{t-1} + \cdots + (-1)^t \binom{s}{0}.$$

Lemma 1.

$$T_t^s = \binom{s-1}{t}.$$

Proof. Induction on t :

$$T_t^s = \binom{s}{t} - T_{t-1}^s = \binom{s}{t} - \binom{s-1}{t-1} = \binom{s-1}{t}. \quad \blacksquare$$

Lemma 2. For $j \geq 1$,

$$\binom{j-1}{0} + \binom{j}{1} + \cdots + \binom{j+i-1}{i} = \binom{j+i}{i}.$$

Proof. Induction on i :

$$\binom{j-1}{0} + \binom{j}{1} + \cdots + \binom{j+i-1}{i} = \binom{j+i-1}{i-1} + \binom{j+i-1}{i} = \binom{j+i}{i}. \quad \blacksquare$$

Lemma 3.

$$\sum_{i=0}^l (-1)^i \binom{i+j}{i} \binom{r}{l-i} = \binom{r-j-1}{l}.$$

Proof. Induction on j . If $j = 0$, then

$$\sum_{i=0}^l (-1)^i \binom{i+j}{i} \binom{r}{l-i} = \sum_{i=0}^l (-1)^i \binom{r}{l-i} = T_l^r = \binom{r-1}{l},$$

as required.

Suppose that $j \geq 1$ and the equation of Lemma 3 is true for any $j' \leq j-1$. Applying Lemmas 1 and 2, we obtain:

$$\begin{aligned} \sum_{i=0}^l (-1)^i \binom{i+j}{i} \binom{r}{l-i} &= \sum_{i=0}^l (-1)^i \left(\binom{j-1}{0} + \binom{j}{1} + \dots + \binom{j+i-1}{i} \right) \binom{r}{l-i} \\ &= \binom{j-1}{0} \sum_{i=0}^l (-1)^i \binom{r}{l-i} + \binom{j}{1} \sum_{i=1}^l (-1)^i \binom{r}{l-i} + \dots \\ &\quad + \binom{j+l-1}{l} \sum_{i=l}^l (-1)^i \binom{r}{0} \\ &= \binom{j-1}{0} T_l^r - \binom{j}{1} T_{l-1}^r + \dots + \binom{j+l-1}{l} (-1)^l T_0^r \\ &= \sum_{i=0}^l (-1)^i \binom{j+i-1}{i} T_{l-i}^r \\ &= \sum_{i=0}^l (-1)^i \binom{j+i-1}{i} \binom{r-1}{l-i} \\ &= \binom{r-j-1}{l} \quad (\text{by hypothesis}). \quad \blacksquare \end{aligned}$$

References

- [1] N. Alon, J.H. Spencer, *The Probabilistic Method*, John Wiley and Sons, Inc., 1992.
- [2] V.I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices, *Prikl. Mat. i Programirovanie* 11 (1974) 3–8.
- [3] A. Gagarin, V. Zverovich, A generalised upper bound for the k -tuple domination number, *Discrete Math.* (2007), in press (doi:10.1016/j.disc.2007.07.033).
- [4] J. Harant, M.A. Henning, On double domination in graphs, *Discuss. Math. Graph Theory* 25 (2005) 29–34.
- [5] F. Harary, T.W. Haynes, Double domination in graphs, *Ars Combin.* 55 (2000) 201–213.
- [6] F. Harary, T.W. Haynes, Nordhaus–Gaddum inequalities for domination in graphs, *Discrete Math.* 155 (1996) 99–105.
- [7] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
- [8] R. Klasing, C. Laforest, Hardness results and approximation algorithms of k -tuple domination in graphs, *Inform. Process. Lett.* 89 (2) (2004) 75–83.
- [9] L. Lovász, On the ratio of optimal integral and fractional covers, *Discrete Math.* 13 (1975) 383–390.
- [10] C. Payan, Sur le nombre d'absorption d'un graphe simple, *Cahiers Centre Études Recherche Opér.* 17 (1975) 307–317.
- [11] D. Rautenbach, L. Volkmann, New bounds on the k -domination number and the k -tuple domination number, *Appl. Math. Lett.* 20 (2007) 98–102.