Basic Perfect Graphs and Their Extensions

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Abstract

In this article, we present a characterization of basic graphs in terms of forbidden induced subgraphs. This class of graphs was introduced by Conforti, Cornuéjols and Vušković [3], and it plays an essential role in the announced proof of the Strong Perfect Graph Conjecture by Chudnovsky, Robertson, Seymour and Thomas [2]. Then we apply the Reducing Pseudopath Method [13] to characterize the substitutional closure of the class of basic graphs in terms of forbidden induced subgraphs.

Keywords: *perfect graphs, basic graphs, line graphs, substitutional closure, forbidden induced subgraphs.*

1 Introduction

A *clique* in a graph is a vertex subset that induces a complete subgraph (not necessarily maximal). The *clique number* of a graph G, $\omega(G)$, is cardinality of a largest clique in G. A *(proper) k-coloring* of a graph G is a partition $V_1 \cup V_2 \cup \cdots \cup V_k$ of $V(G)$ into k stable sets where some V_i may be empty. If G has a k-coloring then it is a *k-colorable graph.* Usually, 2-colorable graphs are called *bipartite.* The *chromatic number* of G, $\chi(G)$, is the smallest k such that G is a k-colorable graph. The *complement* of a graph G is a graph $\overline{G}$

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such that $V(\overline{G}) = V(G)$ and two distinct vertices u and v are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. The *stability number* of a graph G, $\alpha(G)$, is equal to the clique number of the complement $\overline{G}$.

A graph G is a *perfect graph* if $\omega(H) = \chi(H)$ for every induced subgraph H of G. A *hole* in a graph is an induced cycle C_n, $n \geq 4$. A hole is *odd* if it has an odd number of vertices. The complement of a hole is called an *antihole*, and the complement of an odd hole is called an *odd antihole*. A graph is called a *Berge graph* if it does not contain any odd holes and odd antiholes as induced subgraphs.

In 1961, Berge proposed the following Strong Perfect Graph Conjecture.

Conjecture 1 (Berge [1]) *A graph is perfect if and only if it is a Berge graph.*

A weaker conjecture that the class of perfect graphs is closed under complementation was proved by Lovász (Corollary 1), and it follows from the next theorem:

Theorem 1 (Lovász [8]) *A graph G is perfect if and only if*

$$|V(H)| \leq \alpha(H)\omega(H) \tag{1}$$

for every induced subgraph H of G.

This result was obtained with the help of the so-called Replication Lemma (Lemma 1). Let u be a vertex of a graph G. We add a new vertex u' that is adjacent to all vertices in the closed neighborhood $N[u]$ of u. The resulting graph $\mathrm{Rep}(G, u)$ is said to be obtained by *replication* of the vertex u.

Lemma 1 (Lovász [8]) *If G is a perfect graph and $u \in V(G)$, then* $\mathrm{Rep}(G, u)$ *is also a perfect graph.*

Corollary 1 (Lovász [8]) *A graph is perfect if and only if its complement is perfect.*

A subclass of perfect graphs called basic graphs plays an essential role in the announced proof of the Strong Perfect Graph Conjecture by Chudnovsky, Robertson, Seymour and Thomas [2]. Recall that the *line graph* $L(G)$ of a graph G is the intersection graph of edges of G, that is $V(L(G)) = E(G)$ and two distinct vertices e and e' are adjacent in $L(G)$ if and only if the edges e and e' of G have a common vertex. A graph H is called a *line graph* if $H = L(G)$ for some graph G. The following notation is used:

- $\mathcal{B}$, the class of all bipartite graphs,
- $\overline{\mathcal{B}}$, the class of complements of all bipartite graphs,
- $\mathcal{LB}$, the class of line graphs of all bipartite graphs, and
- $\overline{\mathcal{LB}}$, the class of complements of line graphs of all bipartite graphs.

Definition 1 (Conforti, Cornuéjols and Vušković [3]) *The class of* basic graphs *is defined as*

$$\mathcal{BASIC} = \mathcal{B} \cup \overline{\mathcal{B}} \cup \mathcal{LB} \cup \overline{\mathcal{LB}}.$$

The class of basic graphs is characterized in terms of forbidden induced subgraphs in the next section. In [2], Chudnovsky, Robertson, Seymour and Thomas extended basic graphs by introducing the fifth class of graphs. This fifth class consists of so-called bicographs that have a simple structure. Since the class of bicographs is not hereditary, their modification of basic graphs cannot be characterized in terms of forbidden induced subgraphs.

A graph is *non-basic* if it is not basic. It is easy to show that all basic graphs are perfect. A *(proper) edge k-coloring* of a graph G is a partition $W_1 \cup W_2 \cup \cdots \cup W_k$ of $E(G)$ into k matchings, where some sets of the partition may be empty. The chromatic index of G, $\chi'(G)$, is the minimal k such that G has an edge k-coloring. As usual, $\Delta(G)$ denotes the maximal vertex degree of G.

Theorem 2 (König [7]) *For every bipartite graph G, $\chi'(G) = \Delta(G)$.*

Corollary 2 *All basic graphs are perfect.*

Proof. By Corollary 1, it is sufficient to show that all bipartite graphs and all line graphs of bipartite graphs are perfect. For a bipartite graph G, we have $\omega(G) = \chi(G) = 2$, hence all bipartite graphs are perfect. It is easy to see that $\chi(L(G)) = \chi'(G)$ and $\omega(L(G)) = \Delta(G)$. Theorem 2 implies that $\omega(L(G)) = \chi(L(G))$, therefore $L(G)$ is a perfect graph. ∎

2 Characterization of Basic Graphs

The graphs $C_3 = K_3$, Claw $= K_{1,3}$ and Diamond along with their complements O_3, coClaw and coDiamond are shown in Figure 1. For a set of graphs Z, a graph G is *Z-free* if it does not contain any graph of Z as an induced subgraph.

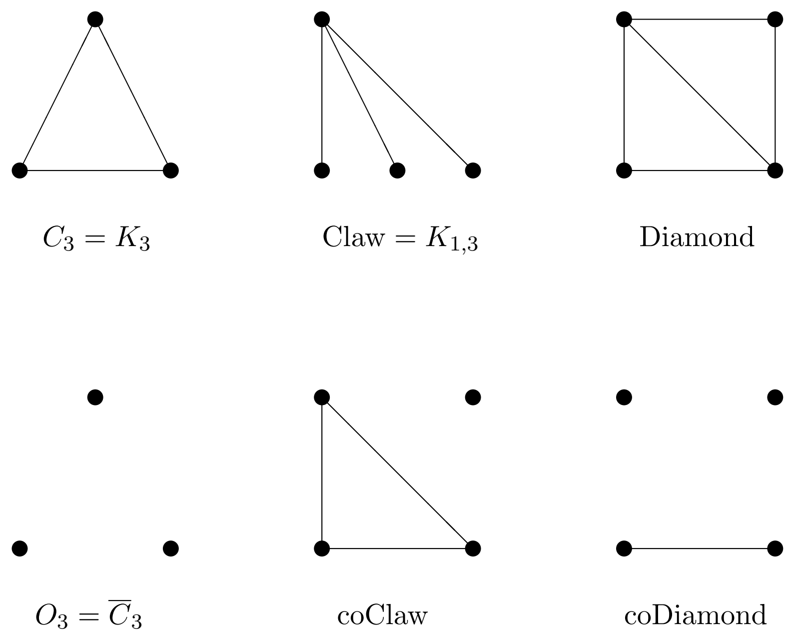

Figure 1

Theorem 3 (König [6]) *A graph is bipartite if and only if it does not contain any odd cycles as induced subgraphs.*

It follows that a Berge graph is bipartite if and only if it is C_3-free. Accordingly, a Berge graph is cobipartite if and only if it is O_3-free. It may be pointed out that a general method for characterization of hereditary classes of line graphs in terms of forbidden induced subgraphs was developed in [4] and [14]. Using this method, it is easy to characterize line graphs of bipartite graphs.

Corollary 3 (Hemminger and Beineke [5], Staton and Wingard [11]) *The class $\mathcal{LB}$ coincides with the class of* (Claw,Diamond,Odd Holes)*-free graphs.*

Corollary 4 *The class $\mathcal{LB}$ is exactly the class of* (Claw, Diamond)*-free Berge graphs, and $\overline{\mathcal{LB}}$ is exactly the class of* (coClaw, coDiamond)*-free Berge graphs.*

The following theorem provides a characterization of basic graphs in terms of forbidden induced subgraphs.

Theorem 4 *A graph G is basic if and only if it does not contain any of the graphs $G_1, G_2, \ldots, G_{16}$ (Figure 2), odd holes and odd antiholes as induced subgraphs.*

Sketch of Proof. It is sufficient to prove that each non-basic Berge graph contains at least one of the graphs $G_1, G_2, \ldots, G_{16}$ (Figure 2) as an induced subgraph. Depending on the existence of Claw and coClaw, we split all non-basic Berge graphs into four subclasses. Table 1 shows the four possible variants for an arbitrary non-basic Berge graph G, where "yes" means that G contains the corresponding induced subgraph and "no" means that G does not.

Table 1

Class	$\mathbf{C_3}$	$\mathbf{O_3}$	**Claw**	**Diamond**	**coClaw**	**coDiamond**
Class 1			yes		yes	
Class 2	yes		yes		no	yes
Class 3		yes	no	yes	yes	
Class 4			no	yes	no	yes

For example, Class 1 consists of all Berge graphs that contain both Claw and coClaw as induced subgraphs. By Corollary 3, Class 1 is disjoint from $\mathcal{LB} \cup \overline{\mathcal{LB}}$. Since O_3 is an induced subgraph of Claw and C_3 is an induced subgraph of coClaw, Class 1 is also disjoint from $\mathcal{B} \cup \overline{\mathcal{B}}$. Thus, Class 1 consists of non-basic Berge graphs only. Class 2 is disjoint from $\mathcal{LB} \cup \overline{\mathcal{B}}$ because Claw is forbidden and O_3 is an induced subgraph of Claw. Since graphs from Class 2 are coClaw-free and Class 2 must be disjoint from $\overline{\mathcal{LB}} \cup \mathcal{B}$, it follows that graphs from Class 2 have to contain both coDiamond and C_3 as induced subgraphs.

Thus, there are four possibilities to consider.

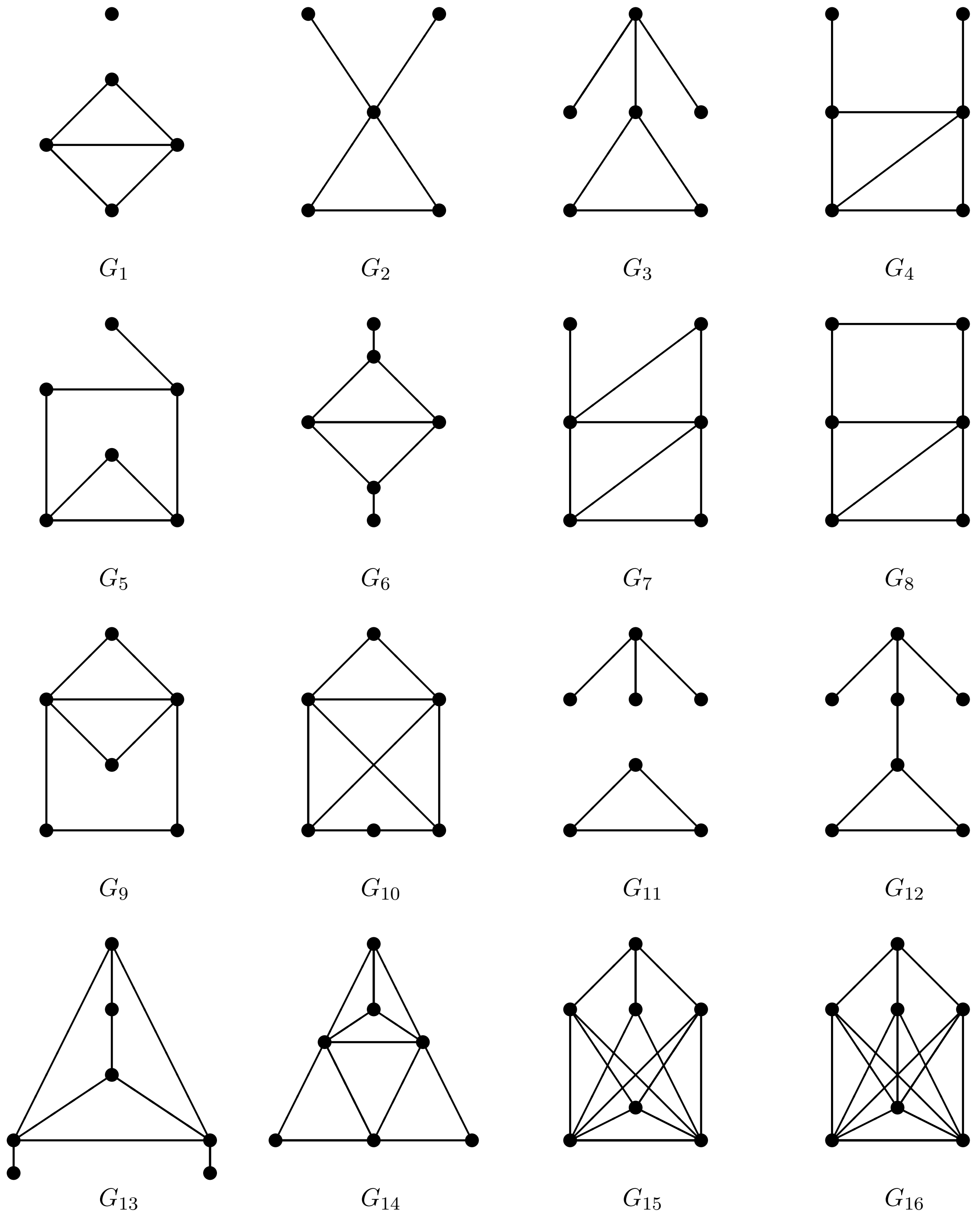

Figure 2. Minimal forbidden induced subgraphs for basic graphs within Berge graphs.

Class 1: If Claw and coClaw have an edge in common, then there are four edges undetermined. Although there are 16 possible combinations, there is no need to consider all the 16 subcases separately. Let us illustrate the technique for this case. Let $V(\text{Claw}) = \{c, a, d, e\}$ and c be the central vertex of Claw, and let $V(\text{coClaw}) = \{a, b, c, i\}$ and i be the isolated vertex of coClaw. Also, let (a, c) be the common edge and H denote the graph induced by those six vertices. Thus, the four edges undetermined in H are $(d, b), (d, i), (e, b)$ and (e, i). The graph $H - i$ is not isomorphic to G_2 in Figure 2. Therefore, we may assume without loss of generality that d is adjacent to b. Now, the vertex d is adjacent to i, for otherwise $H - e$ is isomorphic to G_1. Suppose that e is not adjacent to b. Then H is isomorphic to G_4 or G_8 depending on the existence of (e, i). Hence e is adjacent to b. Now e is adjacent to i, for otherwise $H - d$ is isomorphic to G_1. Thus, H is isomorphic to G_{10}. Therefore, each graph having Claw and coClaw with a common edge contains one of the graphs of Figure 2.

If Claw and coClaw have a pair of non-adjacent vertices in common, then there are four edges undetermined. Suppose that Claw and coClaw have just one vertex in common. Then there are four cases to consider, and each case leads to nine edges undetermined. Finally, if Claw and coClaw are disjoint, then we have to consider graphs of order 8. It can be shown that each graph in the above cases contains one of the graphs of Figure 2.

Class 2: Table 1 implies an upper bound 11 on the maximal order of a minimal graph in Class 2. To reduce the upper bound, we can easily check that each minimal coClaw-free Berge graph that contains C_3 and an induced Claw has five vertices – all such graphs are shown in Figure 3. It follows that the order of a minimal graph in Class 2 is at most nine. It can be shown that each graph in Class 2 contains one of the graphs of Figure 2.

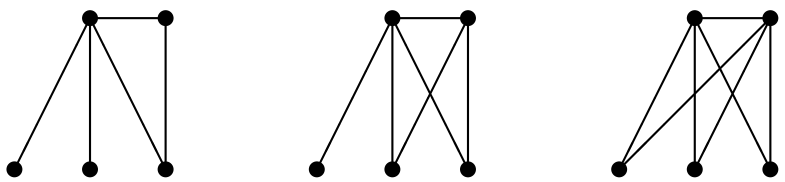

Figure 3. Illustration for Class 2.

Class 3: This class is complementary to Class 2. Therefore, the order of a minimal graph in Class 3 is at most 9 and the result follows.

Class 4: There are two cases to consider if Diamond and coDiamond have an edge in common, and there are two cases to consider if Diamond and coDiamond have a pair of non-adjacent vertices in common. Suppose that Diamond and coDiamond have just one vertex in common. Then there are four cases to consider. The final case when Diamond and coDiamond are disjoint produces graphs of order 8. Since both Claw and coClaw are forbidden as induced subgraphs, it is not very difficult to show that Class 4 is empty. ∎

We only present the sketch of the proof because the actual proof is very long. It may be pointed out that the idea developed above can be used to verify the result of Theorem 4 by a computer – in fact, we carried out a computer search confirming this result.

3 Substitutional Closure

We need the following important definitions.

Definition 2 *Let G and H be graphs. A* substitution of H in G replacing a vertex *$v \in V(G)$ is the graph $G(v \to H)$ consisting of disjoint union of H and $G - v$ with the additional edge-set $\{xy : x \in V(H), y \in N_G(v)\}$.*

For a class of graphs $\mathcal{P}$, its *substitutional closure* $\mathcal{P}^*$ consists of all graphs that can be obtained from $\mathcal{P}$ by repeated substitutions, i.e., $\mathcal{P}^*$ is generated by the following rules:

(S1) put $\mathcal{P} \subseteq \mathcal{P}^*$, and

(S2) if $G, H \in \mathcal{P}^*$ and $v \in V(G)$, then $G(v \to H) \in \mathcal{P}^*$.

Definition 3 *A set $W \subseteq V(G)$ is called* homogeneous *in a graph G if*

(H1) $2 \le |W| \le |V(G)| - 1$, *and*

(H2) $N(x)\backslash W = N(y)\backslash W$ *for each* $x, y \in W$.

According to (H2), a homogeneous set W specifies a partition $W \cup W^+ \cup W^-$ of $V(G)$ such that

- every vertex of W is adjacent to every vertex of W^+, denoted $W \sim W^+$, and
- every vertex of W is non-adjacent to every vertex of W^-, denoted $W \not\sim W^-$.

By (H1), $W^+ \cup W^- \neq \emptyset$ for every homogeneous set W. A simple observation is that if $|V(G)| > 1$, $|V(H)| > 1$ and $v \in V(G)$ then $V(H)$ is a homogeneous set in $G(v \to H)$.

Definition 4 *A graph without homogeneous sets is called* prime*. A graph H is called a* (primal) extension *of a graph G if*

(E1) *G is an induced subgraph of H, and*

(E2) *H is a prime graph.*

The Reducing Pseudopath Method for characterizing the substitutional closure of hereditary classes was introduced in [13] and it is based on the following definition.

Definition 5 *Let G be an induced subgraph of a graph H, and let W be a homogeneous set of G. We define a* reducing W-pseudopath (with respect to G) *in H as a sequence*

$$R = (u_1, u_2, \dots, u_t), \;\; t \ge 1, \tag{2}$$

of pairwise distinct vertices of $V(H)\backslash V(G)$ satisfying the following conditions:

(R1) *there exist vertices $w_1, w_2 \in W$ such that*

> **(R1a)** $u_1 \sim w_1$, *and*
>
> **(R1b)** $u_1 \not\sim w_2$;

(R2) *for each $i = 2, 3, \dots, t$ either*

(R2a) *$u_i \sim u_{i-1}$ and $u_i \not\sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$, or*

(R2b) *$u_i \not\sim u_{i-1}$ and $u_i \sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$ (if $i = 2$, $\{u_1, u_2, \ldots, u_{i-2}\} = \emptyset$);*

(R3) *for every $i = 1, 2, \ldots, t-1$*

(R3a) *$u_i \sim W^+$, and*

(R3b) *$u_i \not\sim W^-$;*

(R4) *either*

(R4a) *$u_t \not\sim x$ for a vertex $x \in W^+$, or*

(R4b) *$u_t \sim y$ for a vertex $y \in W^-$.*

The *length* of a reducing pseudopath (2) is t.

Theorem 5 (Zverovich [13]) *Let H be an extension of its induced subgraph G, and let W be a homogeneous set of G. Then there exists a reducing W-pseudopath with respect to any induced copy of G in H.*

Lovasz' Replication Lemma (Lemma 1) implies that the class of perfect graphs is closed under substitutions of complete subgraphs, i.e., if G is perfect and $v \in V(G)$, then the graph $G(v \to K_n)$ is perfect for every $n \geq 1$.

Theorem 6 *The class of all perfect graphs is closed under substitutions.*

Proof. Suppose that G and H are perfect graphs and $v \in V(G)$. We show that the graph $F = G(v \to H)$ is perfect. We choose a maximum clique K in H, and denote $L = V(H) \backslash K$. By Lemma 1, the induced subgraph $F - L$ is perfect. In particular, we can color $F - L$ with $\omega(F - L) = \omega(F)$ colors. As a result, we color the clique K with $|K| = \omega(H)$ colors. Since H is a perfect graph, we can extend the $|K|$-coloring of K to a $|K|$-coloring of H, thus obtaining an $\omega(F)$-coloring of F.

For each induced subgraph F' of F, we also have $\chi(F') = \omega(F')$. Indeed, either

- F' is an induced subgraph of the perfect graph G, or
- F' is an induced subgraph of the perfect graph H, or
- $F' = G'(v \to H')$, where G' is an induced subgraph of G containing the vertex v and H' is an induced subgraph of H.

In the latter case, both G' and H' are perfect graphs. Therefore, we can use the same proof as above. ∎

In the next section, we shall apply the Reducing Pseudopath Method to the class of basic graphs to produce its extension.

4 Extension of Basic Graphs

Zverovich [12] found some conditions on a homogeneous set W such that there exists a reducing W-pseudopath of a bounded length. In particular, if W induces $P_2, P_3, \overline{P}_2$ or $\overline{P}_3$, then there exists a reducing W-pseudopath of length $t = 1$.

Proposition 1 *Let W be a homogeneous set in a graph G, and let H be an extension of G. If W induces $P_2, P_3, \overline{P}_2$ or $\overline{P}_3$, then there exists a set $Y \subseteq V(H)$ inducing G, and H contains a reducing W-pseudopath (2) with respect to $H(Y)$ having $t = 1$.*

Proof. Let $X \subseteq V(H)$ be a set that induces G in H. By Theorem 5, there exists a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$ with respect to $G = H(X)$ in H. We may assume that R is shortest, i.e., t has the minimum value taken over all induced copies of G in H and all corresponding reducing pseudopaths. Suppose that $t \geq 2$. By (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$ for some $w_1, w_2 \in W$. It is easy to see that there exists a vertex $w \in W$ such that the set $Y = (X \backslash \{w_2\}) \cup \{u_1\}$ induces G. Recall that according to (R3), $u_1 \sim W^+$ and $u_1 \not\sim W^-$, since $t \geq 2$. For example, if the vertices w_1 and w_2 are adjacent, then the set $Y = (X \backslash \{w_2\}) \cup \{u_1\}$ induces G (with u_1 replacing w_2). The condition (R2) implies that $R' = (u_2, u_3, \ldots, u_t)$ is a reducing W-pseudopath with respect to $G = H(Y)$ in H. Since R' is shorter than R, we obtain a contradiction to the minimality of R. ∎

We denote by $\mathcal{BASIC}^*$ the substitutional closure of the class $\mathcal{BASIC}$. All graphs in $\mathcal{BASIC}^*$ are called *superbasic graphs.*

Theorem 7 *The set of all minimal forbidden induced subgraphs for the class $\mathcal{BASIC}^*$ within Berge graphs is*

$$\mathbf{F} = \{F_1, F_2, \ldots, F_8, F_9 = \overline{F}_7, F_{10} = \overline{F}_8, F_{11}, F_{12} = \overline{F}_{11}\} \cup \{S_n, \overline{S}_n : n \geq 1\}$$

shown in Figure 4.

Proof. The fact that all graphs in $\mathbf{F}$ are minimal forbidden induced subgraphs for $\mathcal{BASIC}^*$ can be checked directly. Now let H be an arbitrary minimal forbidden induced subgraphs for $\mathcal{BASIC}^*$. Note that H is a prime graph. We may assume that H is a Berge graph, and $H \notin \mathbf{F}$. Since H is minimal, no graph in $\mathbf{F}$ is an induced subgraph of H. Clearly, H is not a basic graph. Then Theorem 4 implies that at least one of the graphs $G_1, G_2, \ldots, G_{16}$ (Figure 2) is an induced subgraph of H. Note that $\mathbf{F}$ is a self-complementary set, and so is the class of all Berge graphs. In particular, if a graph G cannot be an induced subgraph of H, then the complement of G cannot be an induced subgraph of H either.

Claim 1 *The graph H does not contain both G_4 and G_5 (Figure 2) as induced subgraphs.*

Proof. Indeed, the graphs G_4 and G_5 are included into $\mathbf{F}$ as F_1 and F_2, respectively. ∎

It follows that H does not contain the complements of G_4 and G_5 as induced subgraphs, which are isomorphic to the graphs F_4 and F_5 in $\mathbf{F}$.

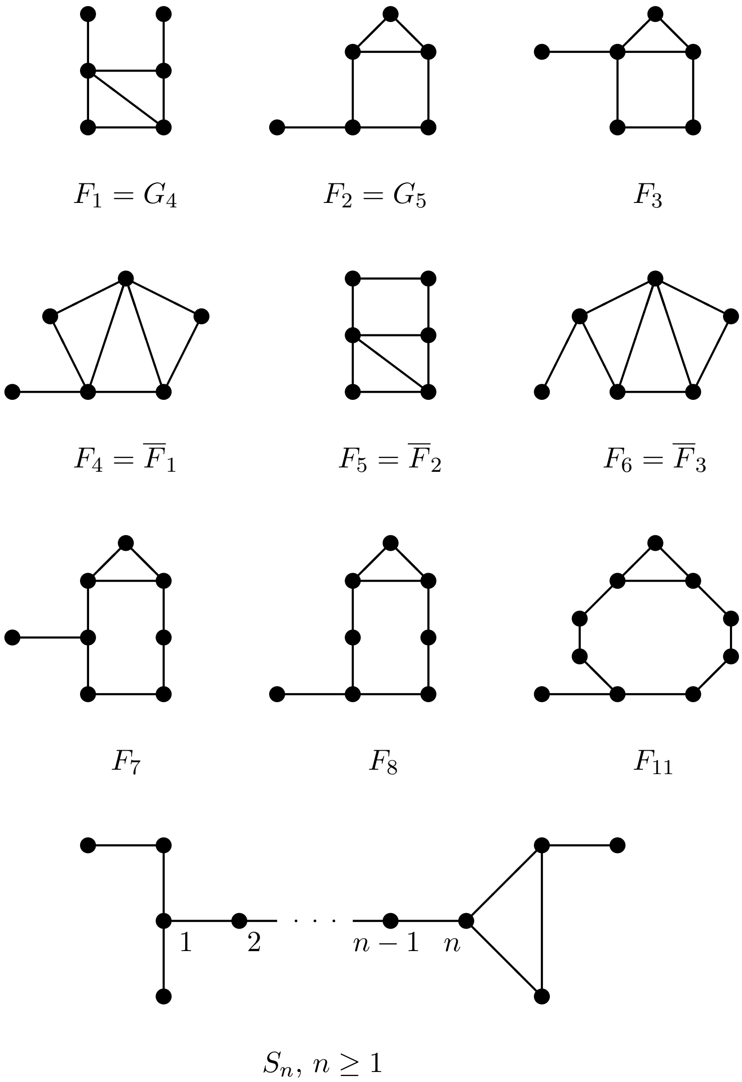

Figure 4. The set **F** ($F_9 = \overline{F}_7$, $F_{10} = \overline{F}_8$, $F_{12} = \overline{F}_{11}$ and $\overline{S}_n$, $n \geq 1$, are not shown).

Now we consider the graph G_6 that has a unique homogeneous set.

Claim 2 *The graph H does not contain both G_6 and G_9 (Figure 2) as induced subgraphs.*

Proof. Suppose that G_6 is an induced subgraph of H. The unique homogeneous set $W = \{w_1, w_2\}$ of G_6 is shown in Figure 5. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H. According to (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$. We have three possibilities to consider. If u_1 is non-adjacent to both a and c, then we delete the vertex d and obtain either F_1 or F_5, a contradiction. Suppose that u_1 is adjacent to a and non-adjacent to c. Then u_1 is non-adjacent to d, for otherwise the set $\{u_1, d, c, w_2, a\}$ induces C_5, a contradiction, since H is a Berge graph. Now we delete b and obtain F_6, a contradiction. If u_1 is adjacent to both a and c, then $t = 1$ and (R4) imply that u_1 is adjacent to a vertex of $W^- = \{b, d\}$, say to b, and deleting a produces either F_1 or F_4, a contradiction. Thus, G_6 cannot be an induced subgraph of H. The result for $G_9 = \overline{G}_6$ follows immediately. ∎

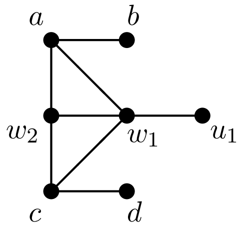

Figure 5. The graph G_6 and a reducing W-pseudopath (u_1).

The graph G_3 is more complicated, since it has two homogeneous sets.

Claim 3 *The graph H does not contain both G_3 and G_{10} (Figure 2) as induced subgraphs.*

Proof. Suppose that G_3 is an induced subgraph of H. The two homogeneous sets $W = \{w_1, w_2\}$ and $X = \{x_1, x_2\}$ of G_3 are shown in Figure 6. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H and there exists a reducing X-pseudopath $R' = (u'_1)$ in H. According to (R1), $u_1 \sim w_1$, $u_1 \not\sim w_2$, $u'_1 \sim x_1$, and $u'_1 \not\sim x_2$. Suppose that $u_1 = u'_1$. We only need to specify edges between u_1 and $\{a, b\}$: if $u \not\sim \{a, b\}$, then $H(u_1, x_1, b, a, w_1)$ is isomorphic to C_5, a contradiction. Otherwise, we delete x_2 and obtain one of F_2, F_3 or F_4, a contradiction. Therefore, $u_1 \neq u'_1$. We separately consider the variants for the induced subgraphs $H(V(G_3) \cup \{u_1\})$ and $H(V(G_3) \cup \{u'_1\})$. Then we compile the results together.

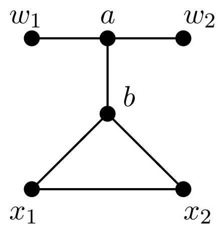

Figure 6. The graph G_3.

Let us consider the graph $H(V(G_3) \cup \{u_1\})$. The condition $t = 1$ and (R4) imply that either u_1 is non-adjacent to a vertex of $W^+ = \{a\}$, or u_1 is adjacent to a vertex of $W^- = \{a, x_1, x_2\}$. Note that u_1 can be adjacent to both x_1 and x_2 or to none of them, since $u_1 \neq u'_1$. We delete the vertex x_2. As a result, we obtain either one of the forbidden induced subgraphs F_1, F_2, F_3, F_4, or a graph containing C_5, or one of the graphs A_1, A_2 of Figure 7.

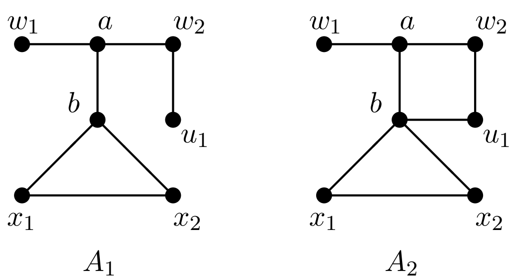

Figure 7. The variants A_1 and A_2.

Now we consider the graph $H(V(G_3) \cup \{u_1'\})$. The condition $t = 1$ and (R4) imply that either u_1' is non-adjacent to a vertex of $X^+ = \{b\}$, or u_1' is adjacent to a vertex of $X^- = \{a, w_1, w_2\}$. Note that u_1' can be adjacent to both w_1 and w_2 or to none of them, since $u_1 \neq u_1'$. We delete the vertex w_2. As a result, we obtain either one of the forbidden induced subgraphs F_2, F_5, F_6, or a graph containing C_5, or one of the graphs B_1, B_2, B_3 of Figure 8.

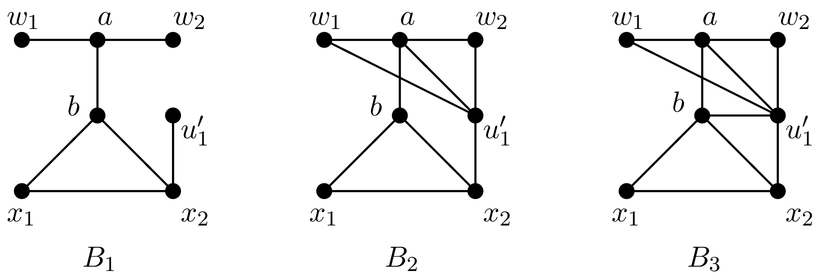

Figure 8. The variants B_1, B_2 and B_3.

Now we consider all combinations (A_i, B_j), $i = 1, 2$ and $j = 1, 2, 3$. Note that each of the six combinations has two variants depending on adjacency of u_1 and u_1'. The combination (A_1, B_1) yields one of the forbidden graphs F_7 or S_2; (A_1, B_2) contains one of the induced subgraphs F_1 or F_4; (A_1, B_3) contains F_1 or F_4; (A_2, B_1) contains F_2 or S_1; (A_2, B_2) contains F_2, and (A_2, B_3) contains F_4. Thus, all the combinations produce a contradiction. The result for $G_{10} = \overline{G}_3$ is straightforward. ∎

The graph G_1 has several homogeneous sets, the largest being $W = \{w_1, w_2, w_3, w_4\}$ (Figure 9). First we reduce G_1 to a set of graphs that have simpler structures of homogeneous sets.

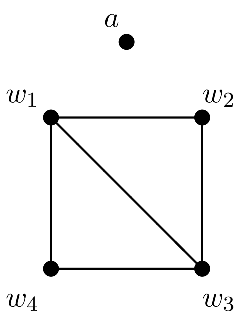

Figure 9. The graph G_1.

Claim 4 *Suppose that H contains G_1 or $G_2 = \overline{G}_1$ as an induced subgraph. Then at least one of $H_1, H_2, \ldots, H_8$ (Figure 10) or their complements is an induced subgraph of H.*

Proof. Suppose that G_1 is an induced subgraph of H. We consider the homogeneous set $W = \{w_1, w_2, w_3, w_4\}$ of G_1 shown in Figure 9. By Theorem 5, there exists a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$ in H. We may assume that R is the shortest reducing W-pseudopath over all induced copies of G_1 in H. According to (R1), u_1 is adjacent to a vertex of W, and u_1 is non-adjacent to a vertex of W. Due to symmetry, there are seven possibilities for adjacencies u_1 and the vertices of W, namely

Possibility 1: $N(u_1) \cap W = \{w_1\}$,

Possibility 2: $N(u_1) \cap W = \{w_2\}$,

Possibility 3: $N(u_1) \cap W = \{w_2, w_4\}$,

Possibility 4: $N(u_1) \cap W = \{w_1, w_2\}$,

Possibility 5: $N(u_1) \cap W = \{w_1, w_3\}$,

Possibility 6: $N(u_1) \cap W = \{w_1, w_2, w_3\}$,

Possibility 7: $N(u_1) \cap W = \{w_1, w_2, w_4\}$.

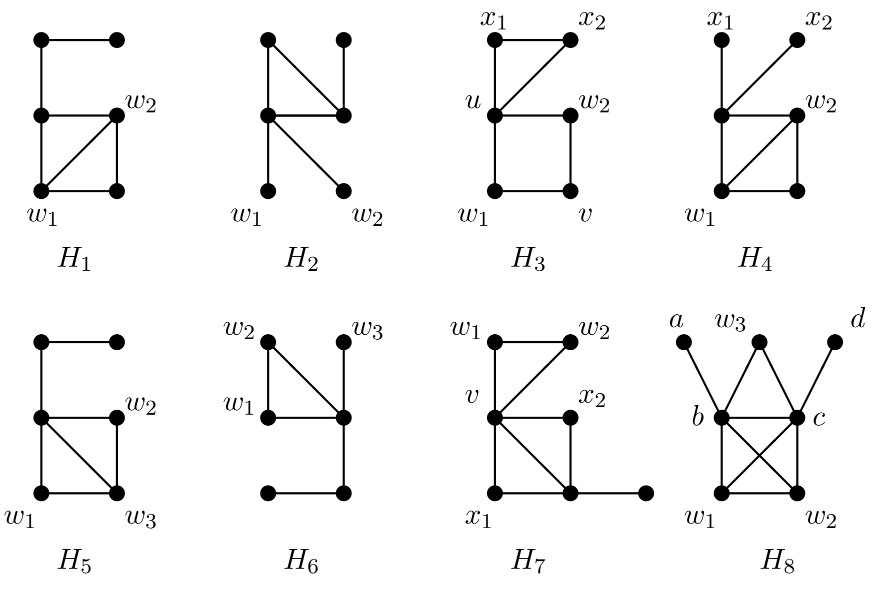

Figure 10. The graphs $H_1, H_2, \ldots, H_8$.

Case 1: $t = 1$. By (R4), the vertex u_1 must be adjacent to a. It can be easily checked that Possibilities 1-7 produce the graphs H_5, H_1, $\overline{H}_3$, F_6, $\overline{H}_4$, $\overline{H}_2$ and $\overline{H}_6$, respectively, and the result follows.

Case 2: $t \geq 2$ in Possibilities 4, 5, 6, 7. By (R3b), u_1 is non-adjacent to the vertex $a \in W^-$. We show that each of Possibilities 4, 5, 6, 7 produces a contradiction to minimality of R. Indeed, we can replace an appropriate vertex $w \in W$ by u_1 and obtain a new induced copy of G_1 with $R' = (u_2, u_3, \dots, u_t)$ such that R' is a shorter reducing W'-pseudopath, where $W' = (W \backslash \{w\}) \cup \{u_1\}$.

In Cases 3 and 4, we use the fact that the condition (R2) determines exactly two variants for adjacencies of each vertex u_i ($i \geq 2$) with the set $W \cup \{u_1, u_2, \dots, u_{i-1}\}$.

Case 3: $t = 2$ in Possibilities 1, 2, 3. We can easily construct all graphs corresponding to Possibilities 1, 2, 3 with $t = 2$: Possibility 1 produces H_5 and $\overline{H}_8$, Possibility 2 produces H_1 and F_1, and Possibility 3 produces $\overline{H}_3$ and $\overline{H}_7$.

Case 4: $t \geq 3$ in Possibilities 1, 2, 3. Assume that the vertex u_2 satisfies (R2a), i.e. u_2 is adjacent to u_1 but it is not adjacent to all the vertices in W. Then Possibilities 1, 2, 3 produce H_5, H_1 and $\overline{H}_3$. Suppose that u_2 satisfies (R2b), i.e. u_2 is non-adjacent to u_1, but u_2 is adjacent to all the vertices in W. Then we can replace the vertex $w_1 \in W$ by u_2 and obtain a new copy of G_1 in H. With respect to the new copy of G_1, we have a shorter reducing W'-pseudopath $R' = (u_3, u_4, \dots, u_t)$ where $W' = (W \backslash \{w_1\}) \cup \{u_2\}$, a contradiction to minimality of R. ∎

By Claim 4, the variant of an induced G_1 or G_2 is reducible to the graphs $H_1, H_2, \dots, H_8$ of Figure 10. We shall show that all of them are impossible.

Claim 5 *The graph H does not contain H_1, H_2 or their complements as induced subgraphs.*

Proof. Suppose that H_1 is an induced subgraph of H. The unique homogeneous set $W = \{w_1, w_2\}$ of H_1 is shown in Figure 10. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H. According to (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$. It is easy to check that we obtain at least one of the graphs F_1, F_4, F_5, F_6 or C_5, a contradiction.

Let H contain H_2 as an induced subgraph. Again, H_2 has a unique homogeneous set, namely $W = \{w_1, w_2\}$ (Figure 10). By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H. According to (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$. It is easy to check that we obtain at least one of the graphs F_1, F_2, F_3, F_4, S_1 or C_5, and the result follows. ∎

Claim 6 *The graph H does not contain H_3 or its complement as an induced subgraph.*

Proof. Suppose that H_3 is an induced subgraph of H. The two homogeneous sets $W = \{w_1, w_2\}$ and $X = \{x_1, x_2\}$ of H_3 are shown in Figure 10. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H and there exists a reducing X-pseudopath $R' = (u'_1)$ in H. According to (R1), $u_1 \sim w_1$, $u_1 \not\sim w_2$, $u'_1 \sim x_1$, and $u'_1 \not\sim x_2$. If $u_1 = u'_1$, then the removal of one of the vertices v, w_1 or x_1 produces F_3 or F_5, a contradiction. Therefore, $u_1 \neq u'_1$.

Since (u_1) is not a reducing X-pseudopath, either (a1) $u_1 \not\sim X$ or (a2) $u_1 \sim X$.

(a1) The condition (R4) shows that u_1 is non-adjacent to a vertex of $W^+ = \{u, v\}$. Hence the removal of w_2 or x_2 produces G_3, F_2 or F_3, a contradiction. Note that H does not contain G_3 by Claim 3.

(a2) The case $N(u_1) = \{u, v, w_1, x_1, x_2\}$ will be considered later. In the other three cases, we delete both or one of the vertices w_1 and x_2, and obtain H_1, F_5 or C_5, a contradiction. Note that H does not contain H_1 by Claim 5.

Since (u'_1) is not a reducing W-pseudopath, either (b1) $u'_1 \not\sim W$ or (b2) $u_1 \sim W$.

(b1) The condition (R4) shows that either u'_1 is non-adjacent to $u \in X^+$, or u'_1 is adjacent to $v \in X^-$. Hence the removal of v or w_1 produces H_2 or F_5, a contradiction. Note that H does not contain H_2 by Claim 5.

(b2) The case $N(u_1') = \{u, v, w_1, w_2, x_1\}$ will be considered later. In the other three cases, we delete v or w_1 and obtain $\overline{H}_1$ or F_6, a contradiction. Note that H does not contain $\overline{H}_1$ by Claim 5.

It remains to consider the situation where $N(u_1) = \{u, v, w_1, x_1, x_2\}$ and $N(u_1') = \{u, v, w_1, w_2, x_1\}$. The vertices u_1 and u_1' may or may not be adjacent. The removal of $\{u, x_1\}$ produces either F_6 if u_1 and u_1' are non-adjacent, or $\overline{H}_2$ if u_1 and u_1' are adjacent, a contradiction, since H does not contain $\overline{H}_2$ by Claim 5. ∎

Claim 7 *The graph H does not contain H_4 or its complement as an induced subgraph.*

Proof. Suppose that H_4 is an induced subgraph of H. It is sufficient to consider one of the two homogeneous sets in H_4, namely $X = \{x_1, x_2\}$ (see Figure 10). By Proposition 1, there exists a reducing X-pseudopath $R = (u_1)$ in H. By (R1), $u_1 \sim x_1$ and $u_1 \not\sim x_2$. If u_1 is adjacent to exactly one vertex of W, then we obtain a graph containing F_1 or F_3 as an induced subgraph, a contradiction. If $u_1 \not\sim W$, then either u_1 is non-adjacent to the unique vertex of X^+ or u_1 is adjacent to the unique vertex of $X^- \backslash W$ by the condition (R4) for $t = 1$. Up to symmetry, we have three variants producing F_3 or H_1 or C_5, a contradiction. If $u_1 \sim W$, then we have four variants containing at least one of the graphs F_1, F_2, F_4 or G_{10} as an induced subgraph, a contradiction. ∎

Claim 8 *The graph H does not contain H_5 or its complement as an induced subgraph.*

Proof. Let H_5 be an induced subgraph of H. We consider the maximal homogeneous set of H_5, namely $W = \{w_1, w_2, w_3\}$ (Figure 10). By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ with respect to H_5 in H. It can be easily checked that $V(H_5) \cup \{u_1\}$ contains one of the forbidden induced subgraphs F_1, F_2, F_4, F_5, F_6 or C_5, or an induced G_3, or an induced H_3, unless we have the variants H_5^1 and H_5^2 shown in Figure 11. Both G_3 and H_3 are impossible by Claims 3 and 6. Therefore, it remains to consider the graphs H_5^1 and H_5^2, each of them having $W = \{w_1, w_2\}$ as a unique homogeneous set. A straightforward application of Proposition 1 produces graphs containing at least one of $F_1, F_2, \ldots, F_6$ as an induced subgraph, a contradiction. ∎

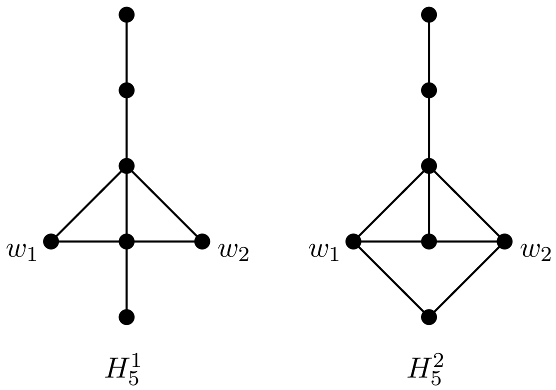

Figure 11. The variants H_5^1 and H_5^2.

Claim 9 *The graph H does not contain H_6 or its complement as an induced subgraph.*

Proof. Let H_6 be an induced subgraph of H. It is sufficient to consider a homogeneous set $W = \{w_1, w_2\}$ (Figure 10) which is not maximal. We apply Proposition 1 and obtain graphs containing at least one of $F_1, F_2, \ldots, F_6$ or H_2, H_5 as an induced subgraph, a contradiction. The result follows from Claims 2 and 9. ∎

Claim 10 *The graph H does not contain H_7 or its complement as an induced subgraph.*

Proof. Suppose that H_7 is an induced subgraph of H. The two homogeneous sets $W = \{w_1, w_2\}$ and $X = \{x_1, x_2\}$ of H_7 are shown in Figure 10. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H and there exists a reducing X-pseudopath $R' = (u_1')$ in H. According to (R1), $u_1 \sim w_1$, $u_1 \not\sim w_2$, $u_1' \sim x_1$, and $u_1' \not\sim x_2$. If $u_1 = u_1'$, then the removal of two appropriate vertices produces F_1 or F_4, a contradiction. Therefore, $u_1 \neq u_1'$.

Since (u_1) is not a reducing X-pseudopath, either (a1) $u_1 \not\sim X$ or (a2) $u_1 \sim X$.

(a1) The condition (R4) implies seven variants for the subgraph induced by $V(H_7) \cup \{u_1\}$. It is not difficult to see that it contains one of F_3, F_4, F_5, H_2 or H_5 as an induced subgraph.

(a2) We obtain induced subgraphs F_5, F_6 or $\overline{H}_1$ unless $N(u_1) = V(H_7)\backslash\{w_2\}$. The latter case is considered later.

Since (u_1') is not a reducing W-pseudopath, either (b1) $u_1' \not\sim W$ or (b2) $u_1 \sim W$.

(b1) The condition (R4) implies seven variants for $H(V(H_7) \cup \{u_1'\})$. It is easy to check that we obtain one of the forbidden induced subgraphs F_1, F_3, F_4, a contradiction.

(b2) In this case, we have forbidden induced subgraphs F_4, F_5, F_6 or H_3, unless either $N(u_1') = V(H_7)\backslash\{x_2\}$ or $N(u_1') = V(H_7)\backslash\{v, x_2\}$ (see Figure 10).

It remains to consider the situation where $N(u_1) = V(H_7)\backslash\{w_2\}$ and either $N(u_1') = V(H_7)\backslash\{x_2\}$ or $N(u_1') = V(H_7)\backslash\{v, x_2\}$. The set $\{w_1, w_2, x_1, x_2, u_1, u_1'\}$ induces either F_2 if u_1 and u_1' are non-adjacent, or F_4 if u_1 and u_1' are adjacent, a contradiction. ∎

Claim 11 *The graph H does not contain H_8 or its complement as an induced subgraph.*

Proof. Suppose that H_8 is an induced subgraph of H. The unique maximal homogeneous set $W = \{w_1, w_2, w_3\}$ of H_8 in shown in Figure 10. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H. According to (R1), u_1 is adjacent to a vertex of W, and u_1 is non-adjacent to a vertex of W. Due to symmetry, we have four possibilities for adjacency u_1 in W. Recall that u_1 must be either adjacent to a vertex of W^- or non-adjacent to a vertex of W^+.

Case 1: $N_H(u_1) \cap W = \{w_1\}$. The removal of two appropriate vertices produces F_1, F_3, F_4 or F_5, a contradiction.

Case 2: $N_H(u_1) \cap W = \{w_3\}$. We obtain one of the forbidden induced subgraphs F_1, F_3, F_4 or F_5 if two vertices are deleted, or we obtain an induced H_7 if a vertex of W^+ is deleted. This contradicts to Claim 9.

Case 3: $N_H(u_1) \cap W = \{w_1, w_3\}$. In this case, we have one of the forbidden induced subgraphs F_3, F_4 or $\overline{H}_1$. By Claim 5, $\overline{H}_1$ cannot be an induced subgraph of H.

Case 4: $N_H(u_1) \cap W = \{w_1, w_2\}$. In this case, we have either one of the forbidden induced subgraphs F_1, F_3, F_4, F_5, G_6, or an induced H_8' shown in Figure 12.

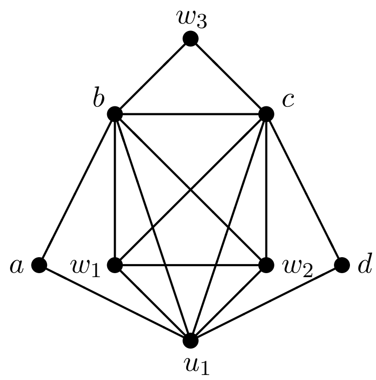

Figure 12. Graph H'_8.

Let us consider the graph H'_8 having $W' = \{w_1, w_2\}$ as a homogeneous set. By Proposition 1, there exists a reducing W'-pseudopath $R' = (u'_1)$ in H. According to (R1), we may assume that u'_1 is adjacent to w_1, and u'_1 is non-adjacent to w_2. If we delete u_1, we essentially obtain Case 1 or Case 4 above (with u'_1 replacing u_1). Therefore, $u'_1 \sim W^+ = \{b, c\}$ and $u'_1 \not\sim W^- = \{a, d\}$. The edges $u'_1 w_3$ and $u'_1 u_1$ are not specified now. If u'_1 and u_1 are non-adjacent, then the set $\{a, b, d, w_1, u_1, u'_1\}$ induces F_4, a contradiction. If u'_1 and u_1 are adjacent, then u'_1 must be adjacent to w_3 according to (R4), and the set $\{b, d, w_1, w_3, u_1, u'_1\}$ induces F_4, a contradiction. ∎

At the moment, we know that H does not contain $G_1, G_2, \ldots, G_{10}$ (Figure 2) as induced subgraphs. Since $G_{14} = \overline{G}_{13}$, $G_{15} = \overline{G}_{12}$ and $G_{16} = \overline{G}_{11}$ (Figure 2), it remains to consider the graphs G_{11}, G_{12} and G_{13}.

Claim 12 *The graph H does not contain G_{12} or its complement as an induced subgraph.*

Proof. Suppose that G_{12} is an induced subgraph of H. The two homogeneous sets $W = \{w_1, w_2\}$ and $X = \{x_1, x_2\}$ of G_{12} are shown in Figure 13. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H and there exists a reducing X-pseudopath $R' = (u'_1)$ in H. According to (R1), $u_1 \sim w_1$, $u_1 \not\sim w_2$, $u'_1 \sim x_1$, and $u'_1 \not\sim x_2$. If $u_1 = u'_1$, then we obtain an induced G_4, G_5, G_7, C_5 or F_8 (Figure 13), a contradiction. Therefore, $u_1 \neq u'_1$. It is easy to check that $N_H(u_1) \cap V(G_{12}) = \{w_1\}$ and $N_H(u'_1) \cap V(G_{12}) = \{x_1\}$, for otherwise one of $G_1, G_2, G_4, G_5, G_7, G_8$ or C_5 is an induced subgraph. If $u_1 \sim u'_1$, then we have an induced C_7, a contradiction. Hence, u_1 and u'_1 are non-adjacent and the set $V(G_{12}) \cup \{u_1, u'_1\}$ induces S_3 (Figure 13). ∎

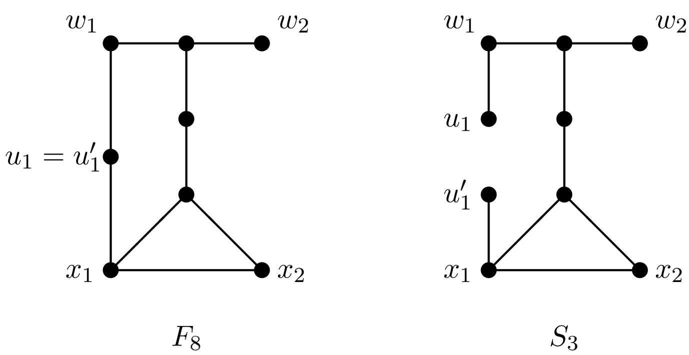

Figure 13. Variants for G_{12}.

Claim 13 *The graph H does not contain G_{13} or its complement as an induced subgraph.*

Proof. A straightforward application of Proposition 1 to the unique homogeneous set of G_{13} produces graphs containing at least one of G_1, G_2, G_5 or G_8 as an induced subgraph, a contradiction. ∎

Now we reduce the most complicated graph G_{11} to a series of graphs having simpler structures of homogeneous sets.

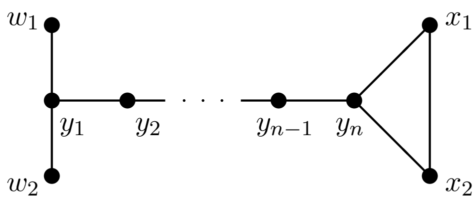

Figure 14. Graphs T_n.

Claim 14 *If H contains G_{11} as an induced subgraph, then at least one of the graphs T_n ($n \geq 4$) shown in Figure 14 is an induced subgraph of H.*

Proof. Let $W \cup X$ induce G_{11} in H, where $W = \{w_1, w_2, w_3\}$ induces a triangle, and $X = \{a, b, c, d\}$ induces a Claw centered at a. By Theorem 5, H contains a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$. We may assume that R is the shortest pseudopath over all induced copies of G_{11} in H. If $t = 1$, then the vertex u_1 is adjacent to a vertex of W and non-adjacent to a vertex of W by (R1). Also, u_1 is adjacent to a vertex of $X = W^-$ by (R4), since $W^+ = \emptyset$. It is easy to check that at least one of G_2, G_3, G_{12} or T_4 is an induced subgraph, and the result follows. Thus, we may assume that $t \geq 2$.

If the vertex u_1 is adjacent to two vertices of W, say w_1 and w_2, then we can delete w_3 and obtain a copy of G_{11} induced by $\{w_1, w_2, u_1\} \cup X$. We have a shorter reducing $\{w_1, w_2, u_1\}$-pseudopath, namely $R' = (u_2, u_3, \ldots, u_t)$, a contradiction to the choice of R. Thus, u_1 is adjacent to exactly one vertex of W.

Now we show that each vertex u_i ($i = 2, 3, \ldots, t-1$) satisfies (R2a), that is $u_i \sim u_{i-1}$ and $u_i \not\sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$. If it does not hold, (R2) implies that there exists $i \in \{2, 3, \ldots, t-1\}$ such that u_i satisfies (R2b). In other words, $u_i \not\sim u_{i-1}$ and $u_i \sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$. Hence we can replace w_3 by u_i and obtain a copy of G_{11} induced by $\{w_1, w_2, u_i\} \cup X$. We have a shorter reducing $\{w_1, w_2, u_i\}$-pseudopath, namely $R_i = (u_{i+1}, u_{i+2}, \ldots, u_t)$, a contradiction to the choice of R. Thus, $(u_1, u_2, \ldots, u_{t-1})$ is a path.

According to (R4), u_t is adjacent to at least one vertex of X. It is easy to see that if u_t satisfies (R2b), then the set $W \cup X \cup \{u_t\}$ induces a subgraph containing G_2, G_3 or G_{12} as an induced subgraph, a contradiction. It follows that u_t satisfies (R2a). In other words, R is an induced path with u_1 adjacent to exactly one vertex of W.

If $N_H(u_t) \cap X$ induces a complete subgraph, then we have an induced subgraph T_n ($n \geq 4$) and the result follows. Otherwise u_t is adjacent to distinct non-adjacent vertices $x, x' \in X$. We can delete $X \backslash \{x, x'\}$ and obtain either an induced G_{12} if $t = 2$ or an induced T_n ($n \geq 4$) if $t \geq 3$. ∎

Accordingly, if H contains $\overline{G}_{11}$ as an induced subgraph, then at least one of the graphs $\overline{T}_n$ $(n \geq 4)$ is an induced subgraph of H.

Claim 15 *The graph H does not contain T_n $(n \geq 4)$ shown in Figure 14 as an induced subgraph.*

Proof. Suppose that H contains an induced subgraph $T = T_n$, $n \geq 4$. We may assume that T has the smallest possible value of $n \geq 4$. First we consider the homogeneous set $W = \{w_1, w_2\}$ in T, see Figure 14. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H. By (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$.

Let us prove that $N_H(u_1) \cap V(T) = \{w_1\}$. The vertex u_1 is non-adjacent to y_1, for otherwise either $H(w_1, w_2, y_1, y_2, u_1) = G_2$ if $u_1 \not\sim y_2$ or $H(w_1, w_2, y_1, y_2, y_3, u_1) \in \{G_4, G_7\}$ if $u_1 \sim y_2$, a contradiction. Now we prove that $u_1 \not\sim \{x_1, x_2, y_n\}$. Note that the vertices y_2 and y_n are non-adjacent, since $n \geq 4$. Assume that u_1 is adjacent to $r \geq 1$ vertices of the triangle $C = \{x_1, x_2, y_n\}$. If $u_1 \sim y_2$ and $r = 1$, then the set $C \cup \{u_1, w_1, y_2\}$ induces G_3. If $u_1 \sim y_2$ and $r \geq 2$, say $u_1 \sim \{x_1, x_2\}$, then $\{u_1, w_1, y_2, x_1, x_2\}$ induces G_2. Further, if $u_1 \not\sim y_2$ and $r = 1$, then the removal of $y_3, y_4, \ldots, y_{n-1}$ produces an induced T_4. Therefore, the minimality of T implies that $n = 4$, and we obtain an induced G_2, C_5 or C_7. Finally, if $u_1 \not\sim y_2$ and $r \geq 2$, say $u_1 \sim \{x_1, x_2\}$, then the set $\{u_1, w_1, w_2, y_1, y_2, x_1, x_2\}$ induces G_{12}. Since all the cases produce a contradiction, we have $u_1 \not\sim C$. Suppose that there exists a maximum $i \in \{3, 4, \ldots, n-1\}$ such that u_1 is adjacent to y_i. The absence of C_5 and G_5 implies $i \geq 4$. It follows that the set $\{u_1, y_{i-1}, y_i, \ldots, y_n, x_1, x_2\}$ induces either G_3 or G_{12} or T_k with $k < n$, a contradiction. Then either $N_H(u_1) \cap V(T) = \{w_1\}$ and the proof is complete, or $N_H(u_1) \cap V(T) = \{w_1, y_2\}$. In the latter case, the removal of $\{w_2, y_1\}$ produces a contradiction to minimality of n.

Now we consider the homogeneous set $X = \{x_1, x_2\}$ in T (Figure 14). By Proposition 1, there exists a reducing X-pseudopath $R' = (u'_1)$ in H. By (R1), $u'_1 \sim x_1$ and $u'_1 \not\sim x_2$.

Let us show that $N_H(u'_1) \cap V(T) = \{x_1\}$. The vertex u'_1 is non-adjacent to y_n, for otherwise either $H(y_i, u_1, y_n, x_1, x_2) = G_1$ if u'_1 is non-adjacent to some y_i, $i \leq n-2$, or we have an induced G_8 if $u_1 \not\sim y_{n-1}$, or the set $\{u'_1, y_{n-3}, y_{n-2}, y_{n-1}, x_2\}$ induces G_1, a contradiction. Now we prove that $u'_1 \not\sim C'$, where $C' = \{w_1, w_2, y_1, y_2\}$. Note that the vertices y_2 and y_n are non-adjacent, since $n \geq 4$. Assuming that u'_1 is adjacent to $r \geq 1$ vertices of the C', we either obtain induced G_3, G_{12}, or u'_1 is adjacent to exactly one vertex c of C' and $c \neq y_1$. Since an induced T_4 appears, it follows that $n = 4$. Then it is easy to see that H has at least one of the forbidden induced subgraphs C_5, C_7 or G_3, a contradiction. Thus, $u'_1 \not\sim C'$. Suppose that there exists a minimum $i \in \{2, 3, \ldots, n-2\}$ such that u'_1 is adjacent to y_i. Then we can easily construct an induced subgraph T_k with $k < n$, contrary to the minimality of n. It follows that either $N_H(u'_1) \cap V(T) = \{x_1\}$ and we are done, or $N_H(u'_1) \cap V(T) = \{x_1, y_{n-1}\}$. In the latter case, we have an induced G_5, a contradiction.

Thus, $N_H(u_1) \cap V(T) = \{w_1\}$ and $N_H(u'_1) \cap V(T) = \{x_1\}$. Therefore, the set $V(T) \cup \{u_1, u'_1\}$ induces either S_n $(n \geq 4)$ if u_1 and u'_1 are non-adjacent, or F_{11} if u_1 and u'_1 are adjacent and $n = 4$, or a graph containing an induced C_9 if u_1 and u'_1 are adjacent and $n = 5$, or a graph containing an induced S_5 if u_1 and u'_1 are adjacent and $n \geq 6$. In the latter case, we delete the vertices $y_4, y_5, \ldots, y_{n-2}$ to obtain S_5. ∎

Since all the graphs G_i in Figure 2 have been considered, Theorem 7 is proved. ∎

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