A Proof of a Conjecture on Irredundance Perfect Graphs

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Abstract

Let $ir(G)$ and $\gamma(G)$ be the irredundance number and the domination number of a graph G, respectively. A graph G is called *irredundance perfect* if $ir(H) = \gamma(H)$, for every induced subgraph H of G. In this article we present a result which immediately implies three known conjectures on irredundance perfect graphs.

Keywords: *irredundance perfect graphs; irredundance number; domination number*

1 Introduction

All graphs will be finite and undirected, without loops and multiple edges. If G is a graph, $V(G)$ denotes the set of vertices in G. We write $u \perp X$ if the vertex u is adjacent to all vertices of the set $X \subseteq V(G)$, and $u \pm X$ if u is adjacent to no vertex of X. Let $N(x)$ denote the neighborhood of a vertex x, and let $\langle X \rangle$ denote the subgraph of G induced by $X \subseteq V(G)$. Also let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$.

A set $X \subseteq V(G)$ *dominates* a set $Y \subseteq V(G)$ if $Y \subseteq N[X]$. In particular, if X dominates $V(G)$, then X is called a *dominating set*. The *independent domination number* $i(G)$ is the cardinality of a minimum independent dominating set of G, and the *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of G. For $x \in X$, the set

$$N[x] - N[X - \{x\}]$$

is called the *private neighborhood* of x and is denoted by $PN(x, X)$, or simply $PN(x)$ if X is clear from the context. If $PN(x, X) = \emptyset$, then x is said to be *redundant* in X. A set X containing no redundant vertex is called *irredundant*. The minimum cardinality taken over all maximal irredundant sets of G is the *irredundance number* $ir(G)$.

It is well known that for any graph G,

$$ir(G) \leq \gamma(G) \leq i(G).$$

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[*] Supported by the Alexander von Humboldt Foundation.

A graph G is called *irredundance perfect* if $ir(H) = \gamma(H)$, for every induced subgraph H of G. A graph G is called *domination perfect* if $\gamma(H) = i(H)$, for every induced subgraph H of G.

There are many interesting results on irredundance perfect graphs [2, 3, 5, 6, 8, 9, 10, 11, 12, 14], a short summary of known results and conjectures connected with P_k-free irredundance perfect graphs is given below. The related classes of graphs such as domination perfect graphs, upper domination perfect graphs and upper irredundance perfect graphs are studied as well. For a short survey on domination perfect graphs, see [15], and for a short survey on upper domination perfect graphs and upper irredundance perfect graphs, see [16]. While the irredundance and domination numbers are equal for irredundance perfect graphs, this is not the case in general. A number of authors [1, 2, 4, 7, 13, 17] investigated the ratio of the irredundance number and the domination number for different classes of graphs.

The first result on irredundance perfect graphs is due to Bollobás and Cockayne.

Theorem 1 (Bollobás and Cockayne [2]) *If a graph G does not have two induced subgraphs isomorphic to P_4 with vertex sequences (a_i, b_i, c_i, d_i), $i = 1, 2$, where $b_1, b_2, c_1, c_2, d_1, d_2$ are distinct and $a_i \notin \{c_1, c_2, d_1, d_2\}$ for $i = 1, 2$, then G is an irredundance perfect graph.*

The following result of Favaron improves Theorem 1, since the graphs forbidden in Theorem 2 belong to the family of forbidden graphs of Theorem 1.

Theorem 2 (Favaron [6]) *If a graph G does not contain the graphs P_6, C_6, $2P_4$ and $G_1 - G_3$ in Figure 1 as induced subgraphs, then G is irredundance perfect.*

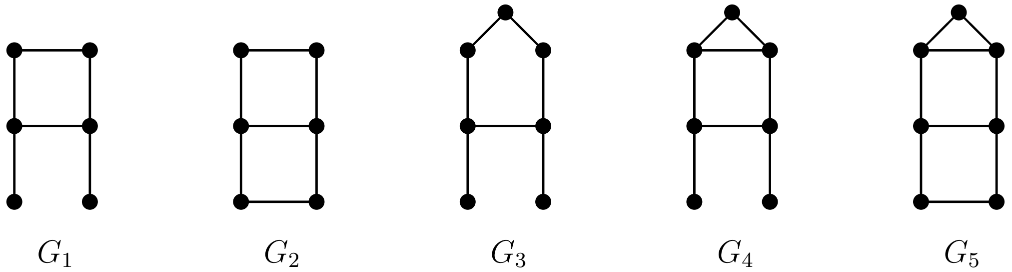

Figure 1.

Favaron conjectured that only three graphs from six forbidden graphs described in Theorem 2 are needed as forbidden subgraphs for an irredundance perfect graph.

Conjecture 1 (Favaron [6], see also [3]) *If a graph G does not contain the graphs P_6 and G_1, G_2 in Figure 1 as induced subgraphs, then G is irredundance perfect.*

The next two conjectures on irredundance perfect graphs are also known.

Conjecture 2 (Faudree, Favaron and Li [5]) *A P_5-free graph is irredundance perfect.*

Conjecture 3 (Puech [12]) *If a graph G does not contain the graphs P_6 and G_4, G_5 in Figure 1 as induced subgraphs, then G is irredundance perfect.*

It is easy to see that Conjecture 3, if true, would imply both Conjecture 1 and Conjecture 2, while Conjectures 1 and 2 do not imply each other. Henning [9] proved Conjecture 1 for a graph G having $ir(H) \leq 4$ for every induced subgraph H of G. Recently, Conjecture 1 was proved in [12] and also in [14]. In fact, a stronger result was proved in [14] that a graph is irredundance perfect if it does not contain P_6, G_1 and G_5 in Figure 1 as induced subgraphs. Conjecture 2 follows from a result of Puech [12] that a graph is irredundance perfect if it does not contain P_6 and H as induced subgraphs, where H is obtained from G_4 in Figure 1 by deleting a vertex of degree 1. In the next section we will prove a theorem which implies all the mentioned conjectures and results.

2 Main Result

Fundamental Statement provides a necessary and sufficient condition for an irredundant set to be maximal, and it will be often used.

Fundamental Statement *Let X be an irredundant set of G, and $U = V(G) - N[X]$. The set X is a maximal irredundant set if and only if for any $v \in N[U]$, the vertex v dominates $PN(x, X)$ for some vertex $x \in X$.*

Proof: Let X be a maximal irredundant set and let $v \in N[U]$. Suppose to the contrary that v does not dominate $PN(x, X)$ for any $x \in X$ and consider the set $X' = X \cup \{v\}$. Since $PN(x, X) \neq \emptyset$ and v does not dominate $PN(x, X)$, we have $PN(x, X') \neq \emptyset$ for any vertex $x \in X$. If $v \in U$, then $v \in PN(v, X')$. If $v \notin U$, then $v \perp u$ for some vertex $u \in U$ and hence $u \in PN(v, X')$. In any case, $PN(v, X') \neq \emptyset$. Thus, the set X' is irredundant in H. This is a contradiction, since X is maximal irredundant.

To prove the sufficiency, suppose that for any $v \in N[U]$, the vertex v dominates $PN(x, X)$ for some $x \in X$. Let $u \in V(G) - X$. If $u \in N[U]$, then u dominates $PN(x, X)$ for some $x \in X$. Therefore, $PN(x, X \cup \{u\}) = \emptyset$. Suppose now that $u \notin N[U]$. We obtain $N[u] \subseteq N[X]$. Consequently, $PN(u, X \cup \{u\}) = \emptyset$. Thus, for any vertex $u \in V(G) - X$, the set $X \cup \{u\}$ is not irredundant. We conclude that X is a maximal irredundant set. ∎

The following theorem gives a proof of Conjecture 3 and also immediately implies Conjectures 1 and 2.

Theorem 3 *If a graph G does not contain the graphs P_6 and G_4, G_5 in Figure 1 as induced subgraphs, then G is an irredundance perfect graph.*

Proof: Let H be a minimum counterexample, i.e., H does not contain P_6, G_4 and G_5 as induced subgraphs, $ir(H) < \gamma(H)$ and $ir(H') = \gamma(H')$ for every proper induced subgraph H' of H. Let X be a maximal irredundant set of H of cardinality $ir(H)$. Denote

$$U = V(H) - N[X],$$

$$PN = \cup_{x \in X} PN(x, X),$$

$$W = V(H) - X - PN - U.$$

Obviously, $|N(w) \cap X| \geq 2$ for any vertex $w \in W$. Also, $U \neq \emptyset$, since X does not dominate H.

The proof of Theorem 3 is based on the following ten lemmas.

Lemma 1 *The graph $\langle X \rangle$ has no isolated vertex.*

Proof: Let v be an isolated vertex in $\langle X \rangle$. Denote $H' = H - N[v]$ and $X' = X - \{v\}$. Obviously, for any vertex $x \in X'$,

$$PN_{H'}(x, X') = PN_H(x, X) \neq \emptyset.$$

Therefore, X' is an irredundant set in H'. Suppose that there is a vertex $u \in V(H') - X'$ such that the set $X' \cup \{u\}$ is irredundant in H', i.e., $PN_{H'}(x, X' \cup \{u\}) \neq \emptyset$ for any vertex $x \in X' \cup \{u\}$. It is not difficult to see that

$$PN_H(x, X \cup \{u\}) = PN_{H'}(x, X' \cup \{u\}) \neq \emptyset$$

for any $x \in X' \cup \{u\}$. Moreover, $PN_H(v, X \cup \{u\}) \neq \emptyset$, since $v \in PN_H(v, X \cup \{u\})$. We conclude that $X \cup \{u\}$ is an irredundant set in H, contrary to the fact that X is maximal irredundant. Consequently, X' is a maximal irredundant set in H' and hence $ir(H') \leq |X'| = ir(H) - 1$. If D is a minimum dominating set of H', then $D \cup \{v\}$ is a dominating set of H. Therefore, $\gamma(H) \leq |D| + 1 = \gamma(H') + 1$. We obtain

$$ir(H') \leq ir(H) - 1 < \gamma(H) - 1 \leq \gamma(H').$$

Thus, $ir(H') < \gamma(H')$, contrary to the minimality of the graph H. ∎

By Fundamental Statement, for every vertex $u \in U$ there is a vertex $x \in X$ such that $PN(x, X) \subseteq N(u)$. Define

$$X_u = \{x \in X : PN(x, X) \subseteq N(u)\}.$$

Let F denote a subset of X of smallest cardinality such that $X_u \cap F \neq \emptyset$ for each $u \in U$.

Lemma 2 *If $f_1, f_2 \in F$, then there exist vertices $y_i \in PN(f_i)$ and $u_i \in U$ $(i = 1, 2)$ such that $u_1 \perp y_1, u_2 \perp y_2$ and $u_1 \pm y_2, u_2 \pm y_1$. Moreover, for $i = 1, 2$, u_i dominates $PN(f_i)$ and u_i does not dominate $PN(f)$ for each $f \in F - f_i$.*

Proof: The minimality of F implies that for $f_1 \in F$ there is a $u_1 \in U$ such that u_1 dominates $PN(f_1)$ and $X_{u_1} \cap (F - f_1) = \emptyset$, i.e., u_1 does not dominate $PN(f)$ for any $f \in F - f_1$. Analogously, for $f_2 \in F$ there is a $u_2 \in U$ dominating $PN(f_2)$ and not dominating $PN(f)$ for any $f \in F - f_2$. Choosing $y_1 \in PN(f_1)$ so that $y_1 \pm u_2$ and choosing $y_2 \in PN(f_2)$ so that $y_2 \pm u_1$, we obtain the desired result. ∎

Put $Z = X - F$, thus $X = F \cup Z$. If L is an arbitrary connected component of the graph $H - W$, then define

$$L_X = V(L) \cap X, \quad L_F = L_X \cap F, \quad L_Z = L_X \cap Z,$$

$$L_{PN} = V(L) \cap PN, \qquad L_U = V(L) \cap U.$$

Obviously,

$$V(L) = L_X \cup L_{PN} \cup L_U.$$

By Fundamental Statement, any vertex of L_U is adjacent to some vertex of L_{PN}. Also, by definition, any vertex of L_{PN} is adjacent to some vertex of L_X. Therefore, $L_X \neq \emptyset$.

Lemma 3 *The graph $\langle L_X \rangle$ is a connected graph.*

Proof: Suppose to the contrary that $\langle L_X \rangle$ is not connected, and let L_1 and L_2 be two connected components of $\langle L_X \rangle$. Consider a shortest path P between L_1 and L_2 in L. The path P has the form $P = (x_1, v_1, ..., v_k, x_2)$, where $x_1 \in L_1$ and $x_2 \in L_2$. Obviously, $v_1 \notin L_1$, $v_1 \in PN(x_1, X)$ and $v_k \notin L_2$, $v_k \in PN(x_2, X)$, and hence $v_1 \neq v_k$ and $k \geq 2$. By Lemma 1, $x_1 \perp x'_1$ and $x_2 \perp x'_2$, where $x'_1, x'_2 \in X$. Clearly $x'_1, x'_2 \in V(L)$ and hence $x'_1 \in L_1$ and $x'_2 \in L_2$. Note that $x'_1 \pm v_1$, since $v_1 \in PN(x_1)$, and also $x'_1 \pm v_i$ for $i > 1$, for otherwise we contradict our choice of P. Analogously, $x'_2 \pm v_i$ for any i, $1 \leq i \leq k$. Now the path $(x'_1, x_1, v_1, ..., v_k, x_2, x'_2)$ contains, evidently, the path P_6 as an induced subgraph, a contradiction. Thus, $\langle L_X \rangle$ is a connected graph. ∎

Now we show that the component L may have only one of basic types defined below. Firstly suppose that the set L_F is independent. If $L_Z = \emptyset$, then we have a contradiction to Lemma 1 or 3, and hence $L_Z \neq \emptyset$. Let there exist a vertex $z \in L_Z$ such that $|N(z) \cap L_F| \geq 2$, in this case L is called a *component of type A*. Let $f_1, f_2 \in N(z) \cap L_F$ and let y_1, y_2, u_1, u_2 be chosen as in Lemma 2. We have $y_1 \perp y_2$, for otherwise $\langle u_1, y_1, f_1, z, f_2, y_2 \rangle \cong P_6$. Let us show that $v \perp f_1$ for an arbitrary vertex $v \in L_X - \{z, f_1, f_2\}$. Suppose that $v \pm f_1$ and consider the shortest $(f_1 - v)$-path P in $\langle L_X \rangle$. Such a path must exist by Lemma 3. The distance between f_1 and f_2 in $\langle L_X \rangle$ is two. So, if $f_2 \in P$, then P has the form $P = (f_1, z', f_2, v_1, ..., v_k = v)$, where $k \geq 1$. We obtain $\langle u_1, y_1, f_1, z', f_2, v_1 \rangle \cong P_6$. Hence $f_2 \notin P$ and P has the form $P = (f_1, v_1, ..., v_k = v)$, where $k \geq 2$. We have $\langle u_2, y_2, y_1, f_1, v_1, v_2 \rangle \cong P_6$. Thus we have proved that $v \perp f_1$. We can show analogously that $v \perp f_2$. Since L_F is independent, we obtain $v \in L_Z$, and hence $L_F = \{f_1, f_2\}$. Thus, $z \perp \{f_1, f_2\}$ for any vertex $z \in L_Z$. We summarize all the facts about the component L of type A.

Type A. $L_F = \{f_1, f_2\}$, $f_1 \pm f_2$, $L_Z \neq \emptyset$ and $z \perp \{f_1, f_2\}$ for any vertex $z \in L_Z$. Further, let y_i, u_i $(i = 1, 2)$ be chosen by Lemma 2, i.e., $u_i \perp y_i$, $y_i \perp f_i$ $(i = 1, 2)$ and $u_1 \pm y_2$, $u_2 \pm y_1$. Also, $y_1 \perp y_2$. Put

$$\begin{aligned} N_1 &= \{u \in L_U : u \perp y_1, u \pm y_2\}, \\ N_2 &= \{u \in L_U : u \pm y_1, u \perp y_2\}, \\ N_{1,2} &= \{u \in L_U : u \perp y_1, u \perp y_2\}. \end{aligned}$$

Lemma 4 *Let L be a component of type A. Then*

(a) The set $\{y_1, y_2\}$ dominates $PN(f_1) \cup PN(f_2) \cup L_U$;

(b) There is no edge between N_1 and N_2, between N_1 and $N_{1,2}$, between N_2 and $N_{1,2}$;

(c) $L_U = N_1 \cup N_2 \cup N_{1,2}$.

Proof: Suppose that there is a vertex y, say $y \in PN(f_1)$, such that $y \pm \{y_1, y_2\}$. By Lemma 2, $y \perp u_1$. We obtain $\langle y, u_1, y_1, y_2, f_2, z\rangle \cong P_6$, where $z \in L_Z$, a contradiction. If there is a vertex $u \in L_U$ such that $u \pm \{y_1, y_2\}$, then $f_1, f_2 \notin X_u$ and hence $X_u \subseteq L_Z \subseteq Z$. Thus, $X_u \cap F = \emptyset$, contrary to the choice of F. To prove the second statement of the lemma, let $v \in N_1$ be adjacent to $u \in N_2 \cup N_{1,2}$. We have $\langle v, u, y_2, f_2, z, f_1\rangle \cong P_6$, where $z \in L_Z$, a contradiction. If $v \in N_2$ is adjacent to $u \in N_{1,2}$, then $\langle v, u, y_1, f_1, z, f_1\rangle \cong P_6$, a contradiction. Since $\{y_1, y_2\}$ dominates L_U, we obtain $L_U = N_1 \cup N_2 \cup N_{1,2}$. ∎

Now suppose that there is a $z \in L_Z$ such that $N(z) \cap L_F = \{f\}$, and L is not a component of type A. In this case we say that the *component L has type B.* Assume that $|L_F| > 1$, i.e., there is an $f' \in L_F$, $f' \neq f$. Further, consider the shortest $(f - f')$-path P in $\langle L_X\rangle$. Such a path must exist by Lemma 3. Recall that $f \pm f'$, since by assumption L_F is independent. If $P = (f, g, f')$, then g is adjacent to $f, f' \in L_F$ and $g \in L_Z$ because L_F is independent. Therefore, $|N(g) \cap L_F| \geq 2$, i.e., L must be a component of type A, a contradiction. Thus, $P = (f, z_1, z_2, ..., z_k = f')$, where $k \geq 3$. Since $f \in F$, there is $u \in U$ such that $u \perp y \in PN(f)$. We obtain $\langle u, y, f, z_1, z_2, z_3\rangle \cong P_6$. Thus, $|L_F| = 1$. We summarize the above facts.

Type B. $L_F = \{f\}$ and $L_Z \neq \emptyset$. Let $y \in PN(f)$. The vertex y dominates L_U. If, to the contrary, there is $u \in L_U$ such that $u \pm y$, then $f \notin X_u$ and hence $X_u \subseteq L_Z \subseteq Z$, i.e., $X_u \cap F = \emptyset$, a contradiction.

Consider now the case $N(z) \cap L_F = \emptyset$ for any $z \in L_Z$. By Lemma 3, $\langle L_X\rangle$ is connected, and so $L_F = \emptyset$. The component of this type is called a *component of type C.*

Type C. $L_F = \emptyset$ and $L_Z = L_X \neq \emptyset$. Also, $L_U = \emptyset$, for otherwise $\emptyset \neq X_u \subseteq L_Z \subseteq Z$ for $u \in L_U$, a contradiction.

Now let L_F be a dependent set, i.e., there are $f_1, f_2 \in L_F$ with $f_1 \perp f_2$. If $|L_X| = 2$, then we say that *L has type D.*

Type D. $L_F = \{f_1, f_2\}$, $f_1 \perp f_2$, and $L_Z = \emptyset$. Let y_i, u_i $(i = 1, 2)$ be chosen by Lemma 2, i.e., $u_i \perp y_i$, $y_i \perp f_i$ $(i = 1, 2)$ and $u_1 \pm y_2$, $u_2 \pm y_1$. Moreover, $PN(f_i) \subseteq N(u_i)$ for $i = 1, 2$. The set $\{y_1, y_2\}$ dominates L. Indeed, any vertex v of $PN(f_1) \cup PN(f_2) \cup L_U$ belongs to $N[L_U]$ and, by Fundamental Statement, v must dominate $PN(f_1)$ or $PN(f_2)$, i.e., v is adjacent to y_1 or y_2.

Lemma 5 *If $|L_X| \geq 3$, $f_1, f_2 \in F$, $f_1 \perp f_2$, and y_1, y_2, u_1, u_2 are chosen as in Lemma 2, then*

(a) $y_1 \perp y_2$ and $u_1 \pm u_2$;

(b) $\langle L_X\rangle$ consists of two graphs R_1 and R_2 connected by the bridge f_1f_2;

(c) f_1 dominates R_1, and f_2 dominates R_2;

(d) $\{u_1, y_1\}$ dominates $\cup_{x \in V(R_2) - f_2} PN(x)$, and $\{u_2, y_2\}$ dominates $\cup_{x \in V(R_1) - f_1} PN(x)$.

Proof: By Lemma 3, $\langle L_X\rangle$ is a connected graph. Since $|L_X| \geq 3$, there is an $x \in L_X - \{f_1, f_2\}$ adjacent to f_1 or f_2, say $x \perp f_1$. Let us consider the case $u_1 \perp u_2$. We have $y_1 \perp y_2$, for otherwise $\langle x, f_1, y_1, u_1, u_2, y_2\rangle \cong P_6$. If $x \pm f_2$, then $\langle x, f_1, f_2, y_2, u_2, u_1\rangle \cong P_6$, and if $x \perp f_2$, then $\langle x, f_1, f_2, y_1, y_2, u_1, u_2\rangle \cong G_5$, a contradiction. Consequently, $u_1 \pm u_2$. We have $y_1 \perp y_2$, for otherwise $\langle u_1, y_1, f_1, f_2, y_2, u_2\rangle \cong P_6$. Also, $x \pm f_2$, for otherwise $\langle x, f_1, f_2, y_1, y_2, u_1, u_2\rangle \cong G_4$, a contradiction. By the same argument, no vertex of X can be adjacent to both f_1 and f_2. Let R_1 be a connected component of the graph $\langle L_X\rangle - f_2$

containing f_1. If f_1 does not dominate R_1, then there exist vertices x_1, x_2 from R_1 such that $\langle u_2, y_2, y_1, f_1, x_1, x_2\rangle \cong P_6$. Therefore, f_1 dominates R_1. In the same way, f_2 dominates the connected component R_2 of $\langle L_X\rangle - f_1$ containing f_2. Thus, $f_1 f_2$ is a bridge in $\langle L_X\rangle$, since no vertex of $\langle L_X\rangle$ can be adjacent to both f_1 and f_2. If $\{u_2, y_2\}$ does not dominate $PN(y)$ for some $y \in V(R_1) - f_1$, then there is $g \in PN(X)$ such that $g \pm \{u_2, y_2\}$ and $\langle u_2, y_2, f_2, f_1, y, g\rangle \cong P_6$, a contradiction. In the same way, we can show that $\{u_1, y_1\}$ dominates $PN(y)$ for any $y \in V(R_2) - f_2$. ∎

If $|R_1| \geq 2$ and $|R_2| \geq 2$, then we obtain a *component of type E*. Suppose that there is $f \in L_F - \{f_1, f_2\}$, say $f \in V(R_1)$. By Lemma 5, $f \perp f_1$ and $x \pm \{f, f_1\}$ for $x \in V(R_2) - f_2$. Applying Lemma 5 to $\{f, f_1\}$, we obtain that x must be adjacent to f or f_1, a contradiction. Thus, $L_F = \{f_1, f_2\}$.

Type E. $|R_1| \geq 2$, $|R_2| \geq 2$, $L_F = \{f_1, f_2\}$, $f_1 \perp f_2$. By Lemma 5, $y_1 \perp y_2$ and $u_1 \pm u_2$. The set $\{y_1, y_2\}$ dominates $PN(z)$ for any $z \in L_Z$. Indeed, if $u_1 \perp g$ for $g \in PN(z)$ and $z \in V(R_2) - f_2$, then $\langle u_1, g, z, f_2, f_1, z'\rangle \cong P_6$, where $z' \in V(R_1) - f_1$, and hence $u_1 \pm PN(z)$ for any $z \in V(R_2) - f_2$. On the other hand, by Lemma 5, $\{u_1, y_1\}$ dominates $PN(z)$ for any $z \in V(R_2) - f_2$. Therefore, y_1 dominates $PN(z)$ for any $z \in V(R_2) - f_2$. Analogously, y_2 dominates $PN(z)$ for any $z \in V(R_1) - f_1$. Also, $\{y_1, y_2\}$ dominates L_U. Let $z_1 \in V(R_1) - f_1$ and $z_2 \in V(R_2) - f_2$. By Lemma 5, $\{f_1, f_2\}$ dominates L_X. Also, $\{f_1, f_2\}$ dominates $PN(f_1) \cup PN(f_2)$. Thus, the set $D = (L_X - \{z_1, z_2\}) \cup \{y_1, y_2\}$ dominates L and $|D| = |L_X|$.

It remains to consider the case when precisely one of the graphs R_1 and R_2 is of size 1. Without loss of generality, let $|R_1| = 1$ and $|R_2| \geq 2$. This case is subdivided into two subcases. The *type F* is defined by the property $L_F = \{f_1, f_2\}$.

Type F. $|R_1| = 1$, $|R_2| \geq 2$, $L_F = \{f_1, f_2\}$, $f_1 \perp f_2$. By Lemma 5, $y_1 \perp y_2$, $u_1 \pm u_2$, $\{u_1, y_1\}$ dominates $PN(z)$ for any $z \in L_Z$, f_2 dominates L_X, and $\deg_{\langle L_X\rangle} f_1 = 1$. Obviously, $\{y_1, y_2\}$ dominates L_U. If there is a $g \in PN(f_1)$ such that $g \pm \{y_1, y_2\}$, then $\langle g, u_1, y_1, y_2, f_2, z\rangle \cong P_6$, where $z \in L_Z$. Therefore, $\{y_1, y_2\}$ dominates $PN(f_1) \cup L_U$.

Finally, let $|L_F| \geq 3$, $f_1, f_2, f_3 \in L_F$. By Lemma 5, $y_1 \perp y_2$, $u_1 \pm u_2$, f_2 dominates L_X, and $\deg_{\langle L_X\rangle} f_1 = 1$. Suppose that $|L_X| \geq 4$ and let $x \in L_X - \{f_1, f_2, f_3\}$. Assume that $f_3 \perp x$, thus $\langle x, f_2, f_3\rangle \cong K_3$. Applying Lemma 5 to $\{f_2, f_3\}$, we obtain that $f_2 f_3$ is a bridge in $\langle L_X\rangle$, a contradiction. Hence $f_3 \pm x$. By Lemma 2, u_1 does not dominate $PN(f_3)$, and let $y_3 \in PN(f_3)$, $y_3 \pm u_1$. We have $y_1 \perp y_3$, for otherwise $\langle u_1, y_1, f_1, f_2, f_3, y_3\rangle \cong P_6$. Now $\langle u_1, y_1, y_3, f_3, f_2, x\rangle \cong P_6$, a contradiction. Therefore $|L_X| = 3$.

Type G. $|R_1| = 1$, $|R_2| = 2$, $L_X = L_F = \{f_1, f_2, f_3\}$, $L_Z = \emptyset$, $f_1 \perp f_2$. By Lemma 5, $y_1 \perp y_2$, $u_1 \pm u_2$, $f_2 \perp f_3$, $f_1 \pm f_3$.

In fact, we have proved the following lemma.

Lemma 6 *Any connected component of the graph $H - W$ has one of the types A–G.*

Now we will construct a set I such that $|I| \leq |X|$ and I dominates the graph $H - W$. If L has type C, then we put $L_Z \subseteq I$. Evidently, L_Z dominates L and $|L_Z| = |L_X|$. For type E we add in I the set D constructed in the definition of this type. The types A, B, D, F and G are subdivided into subtypes. Let the component L be of type A.

Type A1. There is a $w \in W$ dominating $L_U \cup PN(f_2)$. Put $D = (L_X - f_2) \cup \{w\} \subseteq I$. Evidently, D dominates L and $|D| = |L_X|$.

Type A2. There is no $w \in W$ dominating $L_U \cup PN(f_2)$, and for some $z \in L_Z$ the set $\{y_1, y_2\}$ does not dominate $PN(z)$. That is, there is $p \in PN(z)$ and $p \pm \{y_1, y_2\}$. If there is a $u \in N_1$ such that $u \pm p$, then $\langle u, y_1, y_2, f_2, z, p\rangle \cong P_6$. Therefore, p dominates N_1. If there is a $u \in N_2$ such that $u \pm p$, then $\langle u, y_2, y_1, f_1, z, p\rangle \cong P_6$. Therefore, p dominates N_2. Finally, let there exist a $u \in N_{1,2}$ such that $u \pm p$. By Lemma 4, $u \pm u_1$ where $u_1 \in N_1$. We have $\langle u_1, p, z, f_2, y_2, u\rangle \cong P_6$, and hence p dominates $N_{1,2}$. Thus, p dominates L_U. Assume that there is $z' \in L_Z$, $z' \neq z$. We know that $z' \perp \{f_1, f_2\}$, and $u_1 \pm u_2$ by Lemma 4. We obtain $\langle u_1, p, u_2, y_2, f_2, z'\rangle \cong P_6$, and hence $L_Z = \{z\}$. Since $p \pm \{y_1, y_2\}$ and $p \perp u_1$, we have p dominates $PN(z)$ by Fundamental Statement. Put $D = \{f_1, f_2, p\} \subseteq I$. Obviously, D dominates L and $|D| = |L_X| = 3$.

Type A3. There is no $w \in W$ dominating $L_U \cup PN(f_2)$, and the set $\{y_1, y_2\}$ dominates $\cup_{z \in L_Z} PN(z)$. Let there exist a vertex $w \in W$ such that w dominates $L_U \cup PN(z)$ for some $z \in L_Z$ and $w \pm \{f_1, y_1, y_2\}$. Put $D = (L_X - z) \cup \{w\} \subseteq I$. Evidently, D dominates L and $|D| = |L_X|$.

Type A4. There is no $w \in W$ dominating $L_U \cup PN(f_2)$, the set $\{y_1, y_2\}$ dominates $\cup_{z \in L_Z} PN(z)$, and there is no vertex $w \in W$ such that w both dominates $L_U \cup PN(z)$ for some $z \in L_Z$ and satisfies $w \pm \{f_1, y_1, y_2\}$. Put $D = (L_X - \{z, f_2\}) \cup \{y_1, y_2\} \subseteq I$, where $z \in L_Z$. Since $\{y_1, y_2\}$ dominates $L_U \cup L_{PN}$ and $z \perp f_1$, $f_2 \perp y_2$, it follows that D dominates L and $|D| = |L_X|$.

Now let L have type B.

Type B1. The vertex y dominates $PN(z)$ for some $z \in L_Z$. Let $p \in PN(z)$. Put $D = (L_X - z) \cup \{y\} \subseteq I$. We have, y dominates $L_U \cup PN(z)$, $L_X - z$ dominates $(L_X \cup L_{PN}) - (PN(z) \cup \{z\})$. By Lemma 3, $z \perp v$ for some $v \in L_X$, and so $L_X - \{z\}$ also dominates z. Thus, D dominates L and $|D| = |L_X|$.

Suppose that y does not dominate $PN(z)$ for any $z \in L_Z$. By Lemma 3, $f \perp z$ for some $z \in L_Z$. Denote the set of such components by $\mathcal{B}$, and $\mathcal{D}$ will denote the set of components of type D. We define subtypes B2, B3, B4, B5 and D1, D2 by the following algorithmic procedure running until $\mathcal{B} = \emptyset$.

- Take a component $L \in \mathcal{B}$.
- If there are another component $L' \in \mathcal{B}$ with the corresponding vertices f', y', z' and a vertex $w \in W$ such that w dominates $L_U \cup L'_U \cup PN(z) \cup PN(z') \cup \{f, f'\}$ and $w \pm \{y, z, y', z'\}$, then L is called a *component of type B2* and L' is called a *component of type B3*. We put $\mathcal{B} = \mathcal{B} - \{L, L'\}$ and start this procedure again.
- If there are $L' \in \mathcal{D}$ and a $w \in W$ such that w dominates $L_U \cup L'_U \cup PN(z) \cup \{f, f_1, f_2\}$ and $w \pm \{y, z, y_1, y_2\}$, then L has *type B4* and L' has *type D1*. We put $\mathcal{B} = \mathcal{B} - L$, $\mathcal{D} = \mathcal{D} - L'$ and start the procedure again.
- If there is no component L' as above, then L is called a *component of type B5*. We put $\mathcal{B} = \mathcal{B} - L$ and go on with the procedure.

The components of the resulting set $\mathcal{D}$ are called *components of type D2*.

Type B2 is defined by the above procedure. Put $D = (L_X - z) \cup \{w\} \subseteq I$. Obviously, D dominates L and $|D| = |L_X|$.

Type B3 is defined by the above procedure. Put $D = L'_X \subseteq I$. Although D does not dominate L', the set I will dominate L', since the above vertex $w \in I$ dominates L'_U and L'_X dominates $L'_X \cup L'_{PN}$.

Type B4 is defined by the above procedure. Put $D = (L_X - z) \cup \{w\} \subseteq I$. Obviously, D dominates L and $|D| = |L_X|$.

Type B5 is defined by the above procedure. Put $D = L_Z \cup \{y\} \subseteq I$. By Fundamental Statement, y dominates $PN(x)$ for some $x \in L_X$. The only possibility is that y dominates $PN(f)$. Moreover, y dominates $L_U \cup \{f\}$. Thus, D dominates L and $|D| = |L_X|$.

Type D1 is defined by the above procedure. Put $D = \{f_1, f_2\} \subseteq I$, thus $|D| = |L'_X| = 2$. Although D does not dominate L', the set I will dominate L', since the above $w \in I$ dominates L'_U and $\{f_1, f_2\}$ dominates $L' - L'_U$.

Type D2 is defined by the above procedure. Put $\{y_1, y_2\} \subseteq I$. The set $\{y_1, y_2\}$ dominates L and $|\{y_1, y_2\}| = |L_X| = 2$.

Now we subdivide the types F and G into subtypes.

Type F1. $L_Z = \{z\}$, y_1 dominates $PN(z)$, and there is no vertex $w \in W$ dominating $L_U \cup PN(z)$. Put $D = \{y_1, y_2, f_2\} \subseteq I$. Obviously, D dominates L and $|D| = 3 = |L_X|$.

Type F2. $L_Z = \{z\}$, y_1 dominates $PN(z)$, and there is a $w \in W$ dominating $L_U \cup PN(z)$. Put $D = \{f_1, f_2, w\} \subseteq I$. Clearly, D dominates L and $|D| = 3 = |L_X|$.

Type F3. $L_Z = \{z\}$ and y_1 does not dominate $PN(z)$, i.e., there is an $y \in PN(z)$ with $y \pm y_1$. Since $\{u_1, y_1\}$ dominates $PN(z)$, we obtain $u_1 \perp y$. Put $D = \{y, y_1, y_2\} \subseteq I$. We know that D dominates $L_U \cup PN(f_1) \cup L_X$. Any vertex $g \in PN(f_2)$ is adjacent to u_2 and, by Fundamental Statement, g dominates $PN(x)$ for some $x \in L_X$. Therefore, D dominates $PN(f_2)$. Suppose now that D does not dominate $PN(z)$, i.e., there is a $g \in PN(z)$ and $g \pm y_1$. We have $g \perp u_1$, since $\{u_1, y_1\}$ dominates $PN(z)$. By Fundamental Statement, g dominates $PN(x)$ for $x \in L_X$, i.e., g is adjacent to y, y_1 or y_2, a contradiction. Thus, D dominates L and $|D| = 3 = |L_X|$.

Type F4. $|L_Z| \geq 2$. Put $D = \{u_1, y_1, y_2, f_2\} \subseteq I$. We know that $\{u_1, y_1\}$ dominates $PN(z)$ for any $z \in L_Z$. Also, $\{y_1, y_2\}$ dominates $L_U \cup PN(f_1)$ and f_2 dominates $PN(f_2) \cup L_X$. Thus, D dominates L and $|D| = 4 \leq |L_X|$.

Type G1. The set $\{y_1, y_2\}$ does not dominate $PN(f_3)$. Let $y_3 \in PN(f_3)$ and $y_3 \pm \{y_1, y_2\}$. Put $D = \{y_1, y_2, y_3\} \subseteq I$. Obviously, D dominates $L_U \cup L_X$. Any vertex $g \in L_{PN}$ is adjacent to $u \in L_U$. By Fundamental Statement, g dominates $PN(x)$ for some $x \in L_X$, i.e., g is dominated by D. Thus, D dominates L and $|D| = |L_X|$.

Type G2. The set $\{y_1, y_2\}$ dominates $PN(f_3)$ but it does not dominate L_U. Put $D = \{y_1, y_2, y_3\} \subseteq I$, where $y_3 \in PN(f_3)$. Evidently, D dominates $L_U \cup L_X$. Any vertex $g \in L_{PN}$ is adjacent to $u \in L_U$. By Fundamental Statement, g dominates $PN(x)$ for some $x \in L_X$, i.e., g is dominated by D. Thus, D dominates L and $|D| = |L_X|$.

Type G3. The set $\{y_1, y_2\}$ dominates $PN(f_3) \cup L_U$, and there is no $w \in W$ dominating $PN(f_3) \cup L_U$. The set $\{y_1, y_2\}$ dominates $PN(f_1)$, for otherwise there is a $g \in PN(f_1)$ such that $g \pm \{y_1, y_2\}$ and $\langle g, u_1, y_1, y_2, f_2, f_3 \rangle \cong P_6$. Put $D = \{y_1, y_2, f_2\} \subseteq I$. Obviously, D dominates L and $|D| = |L_X|$.

Type G4. The set $\{y_1, y_2\}$ dominates $PN(f_3) \cup L_U$, and there is a $w \in W$ dominating $PN(f_3) \cup L_U$. Put $D = \{f_1, f_2, w\} \subseteq I$. Clearly, D dominates L and $|D| = |L_X|$.

The set I constructed above dominates the graph $H - W$, since I dominates every connected component of $H - W$. Also, for every component L we added at most $|L_X|$

vertices in I. We obtain $|I| \leq |X| = ir(H) < \gamma(H)$. Consequently, I does not dominate H. Thus, we have proved the following lemma.

Lemma 7 *There exists a vertex* $w^* \in W - N[I]$.

Put $X^* = N(w^*) \cap X$. Obviously, $X^* \cap I = \emptyset$. By the definition of W,

$$|X^*| \geq 2.$$

Lemma 8 *If the set* X^* *contains a vertex of a component of type D2, then* X^* *contains a vertex of a component of type different from D2.*

Proof: Let $w^* \perp f_1$, where $f_1 \in L_F$ and L has type D2. Since $y_1, y_2 \in I$, we have $w^* \pm \{y_1, y_2\}$. Suppose that $w^* \pm \{u_1, u_2\}$ and consider the case $w^* \perp f_2$. If $y_1 \perp y_2$, then $\langle w^*, f_1, f_2, y_1, y_2, u_1, u_2 \rangle \cong G_4$ or G_5 depending on the existence of u_1u_2. Hence $y_1 \pm y_2$. Now, if $u_1 \pm u_2$, then $\langle u_1, y_1, f_1, f_2, y_2, u_2 \rangle \cong P_6$, and if $u_1 \perp u_2$, then $\langle w^*, f_1, y_1, u_1, u_2, y_2 \rangle \cong P_6$. Thus, $w^* \pm f_2$. Since $|X^*| \geq 2$, we obtain $w^* \perp v$, where $v \in L'_X \cap X^*$ for some connected component L' of $H - W$. We have $\langle u_2, y_2, f_2, f_1, w^*, v \rangle \cong P_6$. Thus, $w^* \perp u_1$ or $w^* \perp u_2$. By Fundamental Statement, w^* dominates $PN(x)$ for some $x \in L'_X$, where L' is another connected component of $H - W$. Since $w^* \pm I$, it follows that L' cannot have type D2. By Lemma 1, $x \perp x'$ for some $x' \in L'_X$. If $w^* \pm \{x, x'\}$, then $\langle y_1, f_1, w^*, y, x, x' \rangle \cong P_6$ where $y \in PN(x)$. Therefore, w^* is adjacent to x or x'. Thus, the set $X^* = N(w^*) \cap X$ contains at least one of the vertices $x, x' \in L'_X$ and L' has type different from D2. ∎

Lemma 9 *The set* X^* *contains vertices of at least two components of the graph* $H - W$.

Proof: Suppose to the contrary that X^* only contains vertices of one component L. By Lemma 8, L cannot have type D2. Furthermore, the component L cannot be of type A1–A3, B1–B5, C, D1, F2 or G4, since for any of these types $|L_X - I| = 0$ or 1, while $|X^*| \geq 2$. If L is of type E and $N(w^*) \cap L_X = \{z_1, z_2\}$, then $\langle z_1, w^*, z_2, f_2, y_2, y_1 \rangle \cong P_6$, a contradiction. Thus, L may have one of the types A4, F1, F3, F4, or G1–G3.

Let L have type A4 and $N(w^*) \cap X = \{f_2, z\}$, where $f_2, z \in L_X$. Since $w^* \pm I$, we have $w^* \pm \{f_1, y_1, y_2\}$. If there is a $u \in N_2$ such that $w^* \pm u$, then $\langle u, y_2, y_1, f_1, z, w^* \rangle \cong P_6$. Therefore, w^* dominates N_2. Let there exist a $u \in N_1 \cup N_{1,2}$ such that $w^* \pm u$. By Lemma 4, $u \pm u_2 \in N_2$. We obtain $\langle u, y_1, f_1, z, w^*, u_2 \rangle \cong P_6$. Therefore, w^* dominates $N_1 \cup N_{1,2}$. Thus, w^* dominates L_U. Since $w^* \perp u_1 \in U$, it follows by Fundamental Statement that w^* dominates $PN(x)$ for some $x \in X$. Clearly, $x \neq f_1, f_2$. Suppose that $x \notin L_Z$ and hence $x \in L'_X$, where L' is another component of $H - W$. Let $w^* \perp y$, where $y \in PN(x)$. We have $\langle y_2, y_1, f_1, z, w^*, y \rangle \cong P_6$. Therefore, $x \in L_Z$. Thus, w^* dominates $L_U \cup PN(x)$ where $x \in L_Z$ and $w^* \pm \{f_1, y_1, y_2\}$, i.e., L is a component of type A3, a contradiction.

Now let L have type F1 and $w^* \perp \{f_1, z\}$. Suppose that there is a $u \in L_U$ such that $u \pm w^*$. If $u \perp y_1$ and $u \pm y_2$, then $\langle u, y_1, y_2, f_2, z, w^* \rangle \cong P_6$, and if $u \pm y_1$ and $u \perp y_2$, then $\langle u, y_2, y_1, f_1, w^*, z \rangle \cong P_6$. Finally, if $u \perp \{y_1, y_2\}$, then $\langle u, y_1, y_2, f_1, f_2, w^*, z \rangle \cong G_5$. Thus, w^* dominates L_U. By Fundamental Statement, w^* dominates $PN(x)$ for some $x \in X$. If $x \in L_X$, then w^* dominates $PN(z)$, since $w^* \pm \{y_1, y_2\}$. This is a contradiction because for a component of type F1 there is no vertex of W dominating $L_U \cup PN(z)$.

Therefore, $x \in L'_X$ for some component L' of $H - W$. Let $w^* \perp g$ where $g \in PN(x)$. We have $\langle y_1, y_2, f_2, z, w^*, g\rangle \cong P_6$, a contradiction. Let L be of type F3, and suppose that $w^* \perp u \in L_U$. By Fundamental Statement, w^* dominates $PN(x)$ for $x \in X$. Clearly, $x \notin L_X$, and hence $x \in L'_X$, where L' is another component of $H - W$. We have $w^* \perp g \in PN(x)$. By Lemma 1, $x \perp x' \in L'_X$. Since $N(w^*) \cap X \subseteq L_X$, we obtain $w^* \pm \{x, x'\}$. If $w^* \perp v \in L_X$ and $y = PN(v) \cap I$, then $y \pm w^*$. We obtain $\langle y, v, w^*, g, x, x'\rangle \cong P_6$. Thus, $w^* \pm L_U$. If $w^* \perp \{f_1, f_2\}$, then $\langle w^*, f_1, f_2, y_1, y_2, u_1, u_2\rangle \cong G_4$, and if $w^* \perp \{f_1, z\}$, then $\langle u_2, y_2, y_1, f_1, w^*, z\rangle \cong P_6$. Thus, $w^* \pm f_1$ and $w^* \perp \{f_2, z\}$. Recall that $y \perp u_1$ and $y \pm y_1$. We have $\langle w^*, z, y, u_1, y_1, f_1\rangle \cong P_6$, a contradiction. Let L be of type F4. Since $|N(w^*) \cap L_X| \geq 2$, the vertex w^* is adjacent to some vertex $z \in L_Z$. We obtain $\langle u_1, y_1, y_2, f_2, z, w^*\rangle \cong P_6$, a contradiction.

Let L have type G1 or G2, and suppose that $w^* \perp u \in L_U$. By Fundamental Statement, w^* dominates $PN(x)$ for $x \in L'_X$, where L' is another component of $H - W$. The vertex w^* is adjacent to a vertex from $\{f_1, f_2\}$, say $w^* \perp f_1$. We obtain $\langle y_2, y_1, f_1, w^*, g, x\rangle \cong P_6$, where $g \in PN(x)$. Thus, $w^* \pm L_U$. We have $w^* \perp f_2$, for otherwise $w^* \perp f_3$ and $\langle u_1, y_1, y_2, f_2, f_3, w^*\rangle \cong P_6$. Also, $w^* \pm f_1$, for otherwise $\langle w^*, f_1, f_2, y_1, y_2, u_1, u_2\rangle \cong G_4$. Since $|X^*| \geq 2$, we obtain $w^* \perp f_3$. If L is of type G1, then $u_1 \perp y_3$, for otherwise $\langle u_1, y_1, f_1, f_2, f_3, y_3\rangle \cong P_6$. Now $\langle w^*, f_3, y_3, u_1, y_1, f_1\rangle \cong P_6$, a contradiction. Let L be of type G2 and suppose that $y_1 \pm y_3$. We have $u_1 \perp y_3$, for otherwise $\langle u_1, y_1, f_1, f_2, f_3, y_3\rangle \cong P_6$. We obtain $\langle w^*, f_3, y_3, u_1, y_1, f_1\rangle \cong P_6$, a contradiction. Thus, $y_1 \perp y_3$. Furthermore, $u_2 \pm y_3$, for otherwise $\langle w^*, f_2, f_1, y_1, y_3, u_2\rangle \cong P_6$. Also, $y_2 \perp y_3$, for otherwise $\langle w^*, f_3, y_3, y_1, y_2, u_1\rangle \cong P_6$. Since $\{y_1, y_2\}$ does not dominate L_U, there is a $u_3 \in L_U \cap N(y_3)$ such that $u_3 \pm \{y_1, y_2\}$. Since $\langle u_2, u_3, y_3, f_3, f_2, f_1\rangle \not\cong P_6$, we obtain $u_2 \pm u_3$. Now $\langle w^*, f_2, f_3, y_2, y_3, u_2, u_3\rangle \cong G_4$, a contradiction. Let L have type G3, and $w^* \perp \{f_1, f_3\}, w \pm f_2$. Assume that there is a $u \in L_U$ with $u \pm w^*$. If $u \perp y_1$ and $u \pm y_2$, then $\langle u, y_1, y_2, f_2, f_3, w^*\rangle \cong P_6$. If $u \pm y_1$ and $u \perp y_2$, then $\langle u, y_2, y_1, f_1, w^*, f_3\rangle \cong P_6$. Finally, if $u \perp \{y_1, y_2\}$, then $\langle u, y_1, y_2, f_1, f_2, f_3, w^*\rangle \cong G_5$. Thus, w^* dominates L_U. By Fundamental Statement, w^* dominates $PN(x)$ for some $x \in X$. Obviously, $x \notin \{f_1, f_2\}$, and $x \neq f_3$ by the hypothesis. Therefore, $x \in L'_X$, where L' is another component of $H - W$. We obtain $\langle y_1, y_2, f_2, f_3, w^*, g\rangle \cong P_6$, where $g \in PN(x)$, a contradiction. ∎

Lemma 10 *The set X^* only contains vertices of components of type B5 and D2.*

Proof: Suppose to the contrary that X^* contains a vertex of a component M of type different from B5 and D2. By Lemma 9, X^* contains a vertex of another component L. Since $X^* \cap I = \emptyset$, neither L nor M can have type B3, C, or D1. Let L have type A4 and $x \in X^* \cap M_X$. Obviously, w^* is adjacent to $f_2 \in L_F$ or $z \in L_Z$, and $w^* \pm \{y_1, y_2, f_1\}$. If $w^* \perp f_2$, then $\langle f_1, y_1, y_2, f_2, w^*, x\rangle \cong P_6$, while if $w^* \perp z$, then $\langle y_2, y_1, f_1, z, w^*, x\rangle \cong P_6$. Both cases yield a contradiction. Therefore, neither L nor M may have type A4. Suppose that one of the components L and M, say L, is of type B2. Then there are a component L' of type B3 and a vertex $w \in W$ such that $w \perp \{f, f'\}$ and $w \pm \{z, z'\}$, where $f', z' \in L'_X$. Since $w, f, f', z' \in I$, we obtain $w^* \pm \{w, f, f', z'\}$. We have $\langle w^*, z, f, w, f', z'\rangle \cong P_6$. Hence, neither L nor M can be of type B2. Assume now that L has type B4, thus there are a component L' of type D1 and $w \in W$ such that $w \perp \{f, f_1, f_2\}$ and $w \pm \{z, y_1, y_2\}$, where $f, z \in L_X$, $\{f_1, f_2\} = L'_X$ and $y_1, y_2 \in L'_{PN}$. Since $w, f, f_1, f_2 \in I$, we have $w^* \pm \{w, f, f_1, f_2\}$.

We obtain $w^* \perp y_1$, for otherwise $\langle w^*, z, f, w, f_1, y_1\rangle \cong P_6$. Now $\langle f, z, w^*, y_1, f_1, f_2\rangle \cong P_6$. Thus, neither L nor M can have type B4. Let L have type E and $x \in X^* \cap M_X$. If $w^* \perp z_1$, then $\langle y_2, f_2, f_1, z_1, w^*, x\rangle \cong P_6$, and if $w^* \perp z_2$, then $\langle y_1, f_1, f_2, z_2, w^*, x\rangle \cong P_6$. Hence, neither L nor M may have type E. Thus, the components L and M may have any of the types A1–A3, B1, F1–F4, or G1–G4, and also L may have type B5 or D2.

Let us show that L contains vertices a, b such that $\langle a, b, w^*\rangle_H \cong P_3$ and a, w^* are its endvertices. If L is of type A1, then we put $a = z$, $b = f_2$. If L has type A2 or A3, then we put $a = f_1$ and $b = z$, where $\{z\} = L_X - I$. Obviously, $w^* \pm a$ and $w^* \perp b$. Let L have type B1. Put $a = y$ and $b = p$ if $w^* \perp p$, and put $a = p$ and $b = z$ if $w^* \pm p$. If L has type B5, put $a = z$ and $b = f$. If L has type D2, then we may assume that $w^* \perp f_1$ and we put $a = y_1$, $b = f_1$. Let L have type F1, F2 or F4. Put $a = f_2$ and $b = x$, where $x \in L_X - f_2$ and $x \perp w^*$. Let L have type F3. If $w^* \perp f_i$ ($i = 1$ or 2), then we put $a = y_i$, $b = f_i$, and if $w^* \perp z$, then we put $a = y$, $b = z$. If L is of type G1 or G2, then put $a = y_i$ and $b = f_i$ for $i \in \{1, 2, 3\}$ such that $w^* \perp f_i$. If L has type G3, then $a = f_2$ and $b = f_1$ if $w^* \perp f_1$ or $b = f_3$ if $w^* \perp f_3$. Finally, if L is of type G4, then $a = f_2$, $b = f_3$. We have, $\langle a, b, w^*\rangle \cong P_3$ and the vertices a, b are not adjacent to vertices of M.

Let M be of type A1. We have $\langle a, b, w^*, f_2, z, f_1\rangle \cong P_6$, a contradiction. Suppose that M has type A2 or A3. We know that $w^* \perp z$ and $w^* \pm \{f_1, f_2\}$. We have w^* dominates $PN(f_2)$, for otherwise there is a $g \in PN(f_2)$ such that $g \pm w^*$ and $\langle g, f_2, z, w^*, b, a\rangle \cong P_6$. Since w^* cannot dominate $M_U \cup PN(f_2)$, there is a $u \in M_U$ such that $u \pm w^*$. If M is of type A2, then $w^* \pm p$ and p dominates M_U. We obtain $\langle a, b, w^*, z, p, u\rangle \cong P_6$. Let M be of type A3, thus there is a $w \in W$ such that w dominates M_U and $w \pm \{f_1, y_1, y_2\}$. We have $w \perp z$, for otherwise $\langle w, u_2, y_2, y_1, f_1, z\rangle \cong P_6$. Since $w \in I$, we have $w \pm w^*$. If w is adjacent to a or b, say $w \perp a$, then $\langle f_1, y_1, y_2, u_2, w, a\rangle \cong P_6$. Hence $w \pm \{a, b\}$. We obtain $\langle a, b, w^*, z, w, u\rangle \cong P_6$, a contradiction. Now let M have type B1. We know that $w^* \perp z$ and $w^* \pm \{f, y\}$. If $w^* \perp p$, then $\langle a, b, w^*, p, y, f\rangle \cong P_6$. Hence $w^* \pm p$ and $\langle a, b, w^*, z, p, y\rangle \cong P_6$. Let M have type F1. We have $\langle a, b, w^*, f_1, f_2, y_2\rangle \cong P_6$ if $w^* \perp f_1$, and $\langle a, b, w^*, z, f_2, y_2\rangle \cong P_6$ if $w^* \perp z$. If M is of type F2, then $\langle a, b, w^*, z, f_2, f_1\rangle \cong P_6$. Let M be of type F3. If $w^* \perp f_2$, then $\langle a, b, w^*, f_2, y_2, y_1\rangle \cong P_6$, and if $w^* \pm f_2$, then $\langle a, b, w^*, x, f_2, y_2\rangle \cong P_6$, where $x = f_1$ or z and $x \perp w^*$. If M has type F4, then $w^* \perp x \in M_X - f_2$. We have $\langle a, b, w^*, x, f_2, y_2\rangle \cong P_6$. Let M have type G1 or G2. If $w^* \perp f_2$, then $\langle a, b, w^*, f_2, y_2, y_1\rangle \cong P_6$, and if $w^* \pm f_2$, then $\langle a, b, w^*, x, f_2, y_2\rangle \cong P_6$, where $x = f_1$ or f_2 and $x \perp w^*$. Let M be of type G3. We have $\langle a, b, w^*, x, f_2, y_2\rangle \cong P_6$ where $x = f_1$ or f_3. Finally, if M has type G4, then $\langle a, b, w^*, f_3, f_2, f_1\rangle \cong P_6$. All cases yield a contradiction. ∎

Now we are ready to deduce a final contradiction. By Lemmas 8, 9 and 10, the set X^* contains a vertex of a component L of type B5 and a vertex of a component L' having type B5 or D2. Let L' be of type B5 and y', f', z' be vertices defined in this type. We have $w^* \perp \{f, f'\}$ and $w^* \pm \{y, z, y', z'\}$. The vertex w^* dominates $PN(z)$, for otherwise $\langle z', f', w^*, f, z, g\rangle \cong P_6$ where $g \in PN(z)$ and $g \pm w^*$. Analogously, w^* dominates $PN(z')$. Since $\langle y, f, w^*, f', y'\rangle \cong P_5$, we easily deduce that w^* dominates $L_U \cup L'_U$. Therefore, L is a component of type B2, a contradiction. Now let L' have type D2 and f_1, f_2, y_1, y_2 be chosen as in the definition of this type. We have $w^* \perp f$ and w^* is adjacent to f_1 or f_2, say $w^* \perp f_1$. Evidently, $w^* \pm \{y, z, y_1, y_2\}$. Also, $\langle z, f, w^*, f_1, f_2, y_2\rangle \not\cong P_6$ implies $w^* \perp f_2$. If $g \in PN(z)$ and $g \pm w^*$, then $\langle g, z, f, w^*, f_1, y_1\rangle \cong P_6$. Hence, w^* dominates $PN(z)$. If

$u \in L_U$ and $u \pm w^*$, then $\langle u, y, f, w^*, f_1, y_1 \rangle \cong P_6$. Therefore, w^* dominates L_U. Suppose that there is a $u \in L'_U$ such that $u \pm w^*$. Clearly, $u \perp y_1$ or $u \perp y_2$, say $u \perp y_1$. We have $\langle z, f, w^*, f_1, y_1, u \rangle \cong P_6$. Thus, w^* dominates L'_U. We obtain that L must have type B4. This contradiction completes the proof of Theorem 3. ∎

Acknowledgment The authors thank the referees for valuable suggestions.

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