Printed in: Proceedings of the Poznań-Zielona Góra Conference on Combinatorics, October 2006
(also available online at http://www.pzg.uz.zgora.pl/)

A Proof of Rautenbach-Volkmann's Conjecture on k-Tuple Domination

Vadim Zverovich
Faculty of Computing, Engineering and Mathematical Sciences
University of the West of England
Bristol, BS16 1QY
UK
vadim.zverovich@uwe.ac.uk

16 September 2006

All graphs will be finite and undirected without multiple edges. If G is a graph, then $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices in G, d_i denotes the degree of v_i and $d = \sum_{i=1}^{n} d_i/n$ is the average degree of G. Let $N(x)$ denote the neighbourhood of a vertex x. Also let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. Denote by $\delta(G)$ the minimum degree of vertices of G. Put $\delta = \delta(G)$. A set X is called a dominating set if every vertex not in X is adjacent to a vertex in X. The minimum cardinality of a dominating set of G is the domination number $\gamma(G)$. A set X is called a k-tuple dominating set of G if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. The minimum cardinality of a k-tuple dominating set of G is the k-tuple domination number $\gamma_{\times k}(G)$. The k-tuple domination number is only defined for graphs with $\delta \geq k-1$. It is easy to see that $\gamma(G) = \gamma_{\times 1}(G)$ and $\gamma_{\times k}(G) \leq \gamma_{\times k'}(G)$ for $k \leq k'$. The 2-tuple domination number $\gamma_{\times 2}(G)$ is called the double domination number and the 3-tuple domination number $\gamma_{\times 3}(G)$ is called the triple domination number.

The following fundamental result for the domination number $\gamma(G)$ of a graph G was proved by Alon and Spencer, Arnautov, Lovász and Payan:

Theorem 1 ([1, 2, 5, 6]) *For any graph G,*

$$\gamma(G) \leq \frac{\ln(\delta+1)+1}{\delta+1} n.$$

A similar upper bound for the double domination number was found by Harant and Henning [4]:

Theorem 2 ([4]) *For any graph G with $\delta \geq 1$,*

$$\gamma_{\times 2}(G) \leq \frac{\ln \delta + \ln(d+1) + 1}{\delta} n.$$

Rautenbach and Volkmann posed the following interesting conjecture for the k-tuple domination number:

Conjecture 1 ([7]) *For any graph G with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta-k+2) + \ln\left(\sum_{i=1}^{n} \binom{d_i+1}{k-1}\right) - \ln(n) + 1}{\delta-k+2} n.$$

For $m \leq \delta$, let us define the m-degree $\widehat{d_m}$ of a graph G as follows:

$$\widehat{d_m} = \widehat{d_m}(G) = \sum_{i=1}^{n} \binom{d_i}{m} /n.$$

Note that $\widehat{d_1}$ is the average degree d of a graph and $\widehat{d_0} = 1$. Also, we put $\widehat{d_{-1}} = 0$.

Since

$$\binom{d_i + 1}{k-1} = \binom{d_i}{k-1} + \binom{d_i}{k-2},$$

we see that the above conjecture can be re-formulated as follows:

Conjecture 1′ *For any graph G with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\widehat{d_{k-1}} + \widehat{d_{k-2}}) + 1}{\delta - k + 2} n.$$

It may be pointed out that this conjecture, if true, would generalise Theorem 2 and also Theorem 1 taking into account that $\widehat{d_{-1}} = 0$. Rautenbach and Volkmann proved the above conjecture for the triple domination number:

Theorem 3 ([7]) *For any graph G with $\delta \geq 2$,*

$$\gamma_{\times 3}(G) \leq \frac{\ln(\delta - 1) + \ln(\widehat{d_2} + d) + 1}{\delta - 1} n.$$

The next result generalises all the above theorems, but it is still far from Conjecture 1′.

Theorem 4 ([3]) *For any graph G with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln\left(\sum_{m=1}^{k-1}(k-m)\widehat{d_m} + \epsilon\right) + 1}{\delta - k + 2} n,$$

where $\epsilon = 1$ if $k = 1$ or 2, and $\epsilon = -d$ if $k \geq 3$.

Rautenbach-Volkmann's conjecture is proved in the following theorem:

Theorem 5 ([8]) *For any graph G with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\widehat{d_{k-1}} + \widehat{d_{k-2}}) + 1}{\delta - k + 2} n.$$

Sketch of the Proof: Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with the probability p, $0 \leq p \leq 1$. For $m = 0, 1, ..., k-1$, let us denote

$$B_m = \{v_i \in V(G) - A : |N(v_i) \cap A| = m\}.$$

Also, for $m = 0, 1, ..., k-2$, we denote

$$A_m = \{v_i \in A : |N(v_i) \cap A| = m\}.$$

For each set A_m, we form a set A'_m in the following way. For every vertex in the set A_m, we take $k - m - 1$ neighbours not in A. Such neighbours always exist because $\delta \geq k-1$.

It is obvious that $|A'_m| \le (k-m-1)|A_m|$. For each set B_m, we form a set B'_m by taking $k-m-1$ neighbours not in A for every vertex in B_m. We have $|B'_m| \le (k-m-1)|B_m|$.

We construct the set D as follows:

$$D = A \cup \left(\bigcup_{m=0}^{k-2} A'_m \right) \cup \left(\bigcup_{m=0}^{k-1} B_m \cup B'_m \right).$$

The set D is a k-tuple dominating set, and the expectation of $|D|$ is

$$E(|D|) \le E(|A|) + \sum_{m=0}^{k-2} (k-m-1) E(|A_m|) + \sum_{m=0}^{k-1} (k-m) E(|B_m|).$$

We have

$$E(|A|) = \sum_{i=1}^{n} P(v_i \in A) = pn, \quad E(|A_m|) = p^{m+1}(1-p)^{\delta-m}\widehat{d}_m n, \quad E(|B_m|) = p^m(1-p)^{\delta-m+1}\widehat{d}_m n$$

and

$$E(|D|) \le pn + (1-p)^{\delta-k+2} n \sum_{m=0}^{k-1} (k-m)p^m(1-p)^{k-m-1}(\widehat{d}_{m-1} + \widehat{d}_m).$$

Let us denote $\mu = \delta - k + 2$. Using the inequality $1 - x \le e^{-x}$, we obtain

$$(1-p)^{\delta-k+2} = (1-p)^{\mu} \le e^{-p\mu}.$$

Thus,

$$E(|D|) \le pn + e^{-p\mu} n \Theta,$$

where

$$\Theta = \sum_{m=0}^{k-1} (k-m)p^m(1-p)^{k-m-1}(\widehat{d}_m + \widehat{d}_{m-1}). \tag{1}$$

It is not too difficult to prove that the function $\Theta(p)$ is monotonously increasing in $0 \le p \le 1$. Therefore, (1) implies

$$\Theta \le \widehat{d}_{k-1} + \widehat{d}_{k-2}.$$

We obtain

$$E(|D|) \le pn + e^{-p\mu} n \Theta \le pn + e^{-p\mu} n (\widehat{d}_{k-1} + \widehat{d}_{k-2}).$$

Let us denote

$$f(p) = pn + e^{-p\mu} n (\widehat{d}_{k-1} + \widehat{d}_{k-2}).$$

For $p \in [0,1]$, the function $f(p)$ is minimised at the point $\min\{1, z\}$, where

$$z = \frac{\ln \mu + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2})}{\mu}.$$

If $z > 1$, then $f(p)$ is minimised at the point $p = 1$ and the result easily follows. If $z \le 1$, then

$$E(|D|) \le f(z) = \left(z + \frac{1}{\mu} \right) n = \frac{\ln \mu + \ln(\widehat{d}_{k-1} + \widehat{d}_{k-2}) + 1}{\mu} n.$$

Since the expectation is an average value, there exists a particular k-tuple dominating set of order at most $f(z)$, as required. The proof of Theorem 5 is complete. ∎

References

[1] N. Alon and J.H. Spencer, *The Probabilistic Method*, John Wiley and Sons, Inc., 1992.

[2] V.I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices. *Prikl. Mat. i Programmirovanie* (11)(1974) 3–8.

[3] A. Gagarin and V. Zverovich, A generalised upper bound for the k-tuple domination number. *Discrete Math.* (to appear)

[4] J. Harant and M.A. Henning, On double domination in graphs. *Discuss. Math. Graph Theory* **25** (2005) 29–34.

[5] L. Lovász, On the ratio of optimal integral and fractional covers. *Discrete Math.* **13** (1975) 383–390.

[6] C. Payan, Sur le nombre d'absorption d'un graphe simple. *Cahiers Centre Études Recherche Opér.* **17** (1975) 307–317.

[7] D. Rautenbach and L. Volkmann, New bounds on the k-domination number and the k-tuple domination number. *Applied Math. Letters* **20** (2007) 98–102.

[8] V. Zverovich, The k-Tuple Domination Number Revisited, *Internal Research Report MS-2006-1*, School of Mathematical Sciences, University of the West of England, Bristol, May 2006 (www.cems.uwe.ac.uk/∼vzverovi).