

Line Hypergraphs: A Survey

Dedicated to the Memory of Lev Arkad'evich Kalužnin

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Abstract. The survey is devoted to line graphs and a new multivalued function \mathcal{L} called the line hypergraph. This function generalizes two classical concepts at once, namely the line graph and the dual hypergraph. In a certain sense, line graphs and dual hypergraphs are extreme values of the function \mathcal{L} . There are many publications about line graphs, but our considerations are restricted to papers concerning Krausz' global characterization of line graphs or Whitney's theorem on edge isomorphisms. The survey covers almost all known results on the function \mathcal{L} because they are concentrated around Krausz' and Whitney's theorems. These results provide evidence that the notion of the line hypergraph is quite natural. It enables one to unify the classical theorems on line graphs and to obtain their more general versions in a simpler way.

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1. Terminology

With minor adaptations, we adopt the terminology of Berge [7].

A *hypergraph* is a pair (V, \mathcal{E}) , where V (the *vertex set*) is a finite nonempty set and \mathcal{E} (the *edge family*) is a finite family of nonempty subsets of V . Thus, $\mathcal{E} = (e_i : i \in I)$, where $\emptyset \neq e_i \subseteq V$ and I is a finite set of *indices*. The set of vertices, the family of edges and the set of indices of a hypergraph H are denoted by VH , $\mathcal{E}H$ and IH , respectively. This definition differs from that of Berge [7], where isolated vertices are not permitted, and of Zykov [33], where empty edges are permitted.

The family $\mathcal{E}(v) = \mathcal{E}_H(v) \subseteq \mathcal{E}H$ of edges incident to a vertex $v \in VH$ is called the *star* of the vertex v . Note that $\mathcal{E}(v)$ is a family of edges but not a partial hypergraph induced by $\mathcal{E}(v)$ as in [7]. The family $SH = (\mathcal{E}(v) : v \in VH)$ is called the *family of stars* of H .

Vertices u and v are called *similar* if $\mathcal{E}(u) = \mathcal{E}(v)$.

A hypergraph having no multiple edges is called *simple*. It is convenient to ignore index sets when handling simple hypergraphs. Thus one defines a simple hypergraph as a pair $H = (V, E)$, where V is as above and E (the *edge set*) is a set of nonempty subsets of V . The edge set of H is denoted by EH .

For a finite family $(H_\lambda : \lambda \in \Lambda)$ of simple hypergraphs, the *set-theoretic union* $\bigcup H_\lambda$ is defined as the simple hypergraph H such that

$$VH = \bigcup_{\lambda \in \Lambda} VH_\lambda \quad \text{and} \quad EH = \bigcup_{\lambda \in \Lambda} EH_\lambda.$$

Let H_1 and H_2 be the following hypergraphs:

$$H_k = (V_k, \mathcal{E}_k), \quad \mathcal{E}_k = (e_i^k : i \in I_k), \quad k = 1, 2.$$

An *isomorphism* $(\alpha, \beta): H_1 \rightarrow H_2$ is a pair of bijections $\alpha: V_1 \rightarrow V_2$ and $\beta: I_1 \rightarrow I_2$ such that for any edge $e_i^1 = \{v_1, \dots, v_d\}$, the equality

$$\alpha(e_i^1) = \{\alpha(v_1), \dots, \alpha(v_d)\} = e_{\beta(i)}^2$$

holds. If hypergraphs H and G are isomorphic, then we write $H \cong G$.

If hypergraphs H_1 and H_2 are simple, then their isomorphism is defined more easily. This is a bijection $\alpha: V_1 \rightarrow V_2$ such that $\alpha(e) \in EH_2$ for $e \subseteq VH_1$ if and only if $e \in EH_1$.

A hypergraph H is called an *r-uniform hypergraph* if all its edges have the same degree r . The *complete r-uniform hypergraph* K_n^r is the simple hypergraph of order n such that its edge set coincides with the set of all r -subsets of the vertex set. For $r > 1$, the *clique of rank r* is K_n^r , the *clique of rank 1* is K_n^1 , and the *clique of rank 0* is the one vertex graph without loops. (This definition differs from that of Berge [6].) The vertex set of a clique will be referred to as a clique also.

Let H be a hypergraph. A finite family

$$Q = (Q_\lambda : \lambda \in \Lambda) \tag{1}$$

of cliques Q_λ is called a *clique covering* of H if $H = \bigcup_{\lambda \in \Lambda} Q_\lambda$. The cliques Q_λ are called the *components* of Q . A component is *trivial* if its rank is equal to 0. The minimum number of components taken over all clique coverings of H is called the *clique covering number* $cc(H)$. It is evident that a clique covering of H exists if and only if H is simple.

If (1) is a clique covering of H and $P_\lambda = VQ_\lambda$, then the family of cliques

$$P = (P_\lambda : \lambda \in \Lambda) \tag{2}$$

is a *clique covering* of the vertex set VH .

Suppose that (1) is a clique covering of a hypergraph H , (2) is the corresponding clique covering of the set VH and ϕ is a permutation on the set VH . Put $\phi(P) = (\phi(P_\lambda) : \lambda \in \Lambda)$. Obviously, $\phi(P)$ is a covering of the set VH , but $\phi(P_\lambda)$ may not be a clique. However, if $\phi(P_\lambda)$ is a clique and $\text{rank } \phi(P_\lambda) = \text{rank } P_\lambda$, then we write $\phi(Q) = (\phi(Q_\lambda) : \lambda \in \Lambda)$.

A vertex subset of a hypergraph H having no pair of adjacent vertices is called *independent*. A partition $VH = V_1 \cup \dots \cup V_s$, $s \leq r$ of the vertex set

VH into independent subsets is called a (*strong*) r -coloring. A hypergraph for which there exists an r -coloring is called r -colorable. A hypergraph is called r -chromatic if it is r -colorable but not $(r - 1)$ -colorable.

At last, a hypergraph possesses the *Helly property* if any edge subfamily F such that every two edges from F have nonempty intersection is contained in a star.

2. Line Graphs and the Function ‘Line Hypergraph’

Let H be a hypergraph without isolated vertices. The *line graph* $L(H)$ is the graph whose vertex set $VL(H)$ is the edge set of H , and such that two vertices are adjacent in $L(H)$ if and only if the corresponding edges are adjacent in H .

Harary [14] notices that the concept of the line graph is so natural that it has been introduced under different names by many authors. Whitney [31] was the first to consider line graphs. The term ‘line graph’ was later introduced by Hoffmann [16]. In the book of Berge [6] these graphs are called *representative graphs*.

The class of line graphs of simple graphs has an attractive peculiarity. It is isomorphically complete by Whitney’s theorem on edge isomorphisms [31], i.e., the recognition problem for graph isomorphisms is reduced in polynomial time to the analogous problem for line graphs. On the other hand, using a technique of Alexeev [1, 2], it is not difficult to see that the class of line graphs is one of the ‘minimal classes’ among all infinite hereditary classes of graphs.

To put it more precisely, let P be an infinite hereditary class of labelled graphs and P_n be the set of graphs from P of order n . The class P is called *nontrivial* if it does not coincide with the class of all graphs. Alexeev proved in [2] that for any infinite hereditary class P there exists an *entropy*, denoted by $h(P)$:

$$h(P) = \lim_{n \rightarrow \infty} (\log_2 |P_n|) \binom{n}{2}^{-1}.$$

If the class P is nontrivial, then $h(P) = 1 - 1/k$ for some natural number $k \in \mathbf{N}$. Conversely, for any $k \in \mathbf{N}$ there exists an infinite hereditary class with $h(P) = 1 - 1/k$ [1]. This means that

$$|P_n| = 2^{\binom{n}{2}(1-1/k)+o(n^2)} \quad \text{or} \quad |P_n| = o(2^{\binom{n}{2}}).$$

Thus, a nontrivial infinite hereditary class of graphs ‘almost contains no graphs’. However, a class with entropy 0 contains a ‘lesser number’ of graphs: $|P_n| = 2^{o(n^2)}$.

PROPOSITION 1. *The entropy of the class of line graphs is equal to 0.*

Proof. Let \mathcal{E}_{ij} be the class of graphs whose vertex set can be divided into i cliques and j independent sets. It is proved in [2] that $h(P) = 0$ if and only if

$$\mathcal{E}_{02}, \mathcal{E}_{11}, \mathcal{E}_{20} \not\subseteq P. \tag{3}$$



Figure 1. The graphs (a) $K_{1,3}$ and (b) G .

The graphs depicted in Figure 1 are not line graphs for any simple graph [5], and $K_{1,3} \in \mathcal{E}_{02} \cap \mathcal{E}_{11}$ and $G \in \mathcal{E}_{20}$. Consequently, (3) holds for the class \mathcal{P} of line graphs of simple graphs. \square

Obviously, the line graph $L(H)$ of a hypergraph H can be represented in the following form: $L(H) = \bigcup_{v \in VH} F_v$, where F_v is the clique whose vertex set is the star $\mathcal{E}(v)$ of H . Therefore, H coincides with the 2-section of the dual hypergraph H^* : $L(H) = (H^*)_2$.

If G is a graph without isolated vertices, then $L(G^*) \cong G$. Thus, any graph is the line graph of some hypergraph. In this connection, the problem of representing graphs as line graphs of hypergraphs with prescribed properties arises.

A comparison of the concepts of line graph $L(H)$ and dual hypergraph H^* has prompted the following definition of the line hypergraph as a multivalued function on the set of hypergraphs [26].

Let H be a hypergraph without isolated vertices with $VH = \{v_1, \dots, v_n\}$, and let $1_H = (\deg v_i : 1 \leq i \leq n)$ be the degree sequence of H . Put

$$0_H = (0_{v_i} : 1 \leq i \leq n)$$

where

$$0_{v_i} = \begin{cases} 0 & \text{if } \deg v_i = 1, \\ 2 & \text{if } \deg v_i > 1. \end{cases}$$

Furthermore, let \mathbf{Z}_+^n be the lattice of integer-valued strings $x = (x_1, \dots, x_n)$ $x_i \geq 0$, $1 \leq i \leq n$, with the following order:

$$x \leq y \iff x_i \leq y_i \quad \text{for } 1 \leq i \leq n.$$

At last, let $\mathcal{D}_H = [0_H, 1_H]$ be an interval in \mathbf{Z}_+^n and $\mathcal{D} = (d_1, \dots, d_n) \in \mathcal{D}_H$. For $v_i \in VH$, the clique of rank d_i with the vertex set $\mathcal{E}(v_i)$ is denoted by F_{v_i} . Put $\mathcal{L}_{\mathcal{D}}(H) = \bigcup_{i=1}^n F_{v_i}$. The hypergraph $\mathcal{L}_{\mathcal{D}}(H)$ is called the *line hypergraph of H with respect to the vector \mathcal{D}* [18, 27].

Write \mathcal{D} in the form $\mathcal{D} = (d_v : v \in VH)$, where $d_v = d_i$ for $v = v_i$. Then the previous definition takes the form

$$\mathcal{L}_{\mathcal{D}}(H) = \bigcup_{v \in VH} F_v. \quad (4)$$

Define the function \mathcal{L} on the set of hypergraphs without isolated vertices as $\mathcal{L}(H) = \{\mathcal{L}_{\mathcal{D}}(H) : \mathcal{D} \in \mathcal{D}_H\}$. The function \mathcal{L} is called the *line hypergraph*. Any element from the set $\mathcal{L}(H)$ is called a *line hypergraph of H* .

It is evident that $L(H) = \mathcal{L}_{0_H}(H) = (\mathcal{L}_{\mathcal{D}}(H))_2$ for any $\mathcal{D} \in \mathcal{D}_H$, and that $\mathcal{L}_{1_H}(H)$ is obtained from the dual hypergraph H^* as a result of identifying multiple edges. If H does not contain similar vertices, then $\mathcal{L}_{1_H}(H) = H^*$. It is also evident that $|\mathcal{L}(H)| = 1$ if and only if $1_H = (2, \dots, 2)$.

EXAMPLE 1. For $H = K_3$, we have

$$1_H = (2, 2, 2) = 0_H \quad \text{and} \quad \mathcal{L}(K_3) = L(K_3) = K_3^* = K_3.$$

EXAMPLE 2. Let $H = K_{1,3}$. Since $0_H = (2, 0, 0, 0)$ and $1_H = (3, 1, 1, 1)$, we have $|\mathcal{D}_H| = 16$. However, some pairs of vectors from \mathcal{D}_H yield isomorphic line hypergraphs. So, the function \mathcal{L} takes the eight values shown in Figure 2. These values correspond to the following vectors:

$$\mathcal{D}_1 = 1_H = (3, 1, 1, 1), \quad \mathcal{D}_2 = (3, 1, 1, 0), \quad \mathcal{D}_3 = (3, 1, 0, 0),$$

$$\mathcal{D}_4 = (3, 0, 0, 0),$$

$$\mathcal{D}_5 = (2, 1, 1, 1), \quad \mathcal{D}_6 = (2, 1, 1, 0), \quad \mathcal{D}_7 = (2, 1, 0, 0),$$

$$\mathcal{D}_8 = 0_H = (2, 0, 0, 0).$$

It follows from (4) that $\mathcal{L}_{\mathcal{D}}(H)$ is a simple hypergraph for which the clique family $(F_v : v \in VH)$ is a covering. Put $(F_v : v \in VH) = Q(H, \mathcal{D})$.

THEOREM 1 ([30]). *Let $H_i, i = 1, 2$, be hypergraphs without isolated vertices whose line hypergraphs $G_i = \mathcal{L}_{\mathcal{D}_i}(H_i)$ are isomorphic. Then*

- (1) $H_1 \cong H_2$ if and only if there exists a bijection $\beta: VG_1 \rightarrow VG_2$ such that $\beta(SH_1) = SH_2$, where SH_i is the family of stars of the hypergraph H_i ;
- (2) An isomorphism $\gamma: G_1 \rightarrow G_2$ is induced by an isomorphism $H_1 \rightarrow H_2$ if and only if $\gamma(Q(H_1, \mathcal{D}_1)) = Q(H_2, \mathcal{D}_2)$.

Let G be a simple hypergraph. The *inverse image* $\mathcal{L}^{-1}(G)$ is defined as the set of hypergraphs H such that $G \cong \mathcal{L}_{\mathcal{D}}(H)$ for some $\mathcal{D} \in \mathcal{D}_H$. The aim of further considerations is to describe the set $\mathcal{L}^{-1}(G)$ for given G . The concept of a canonical hypergraph introduced by Berge in [6] plays an important role here.

Let G be a simple hypergraph, \mathcal{A}_G the set of its clique coverings, and $Q = (Q_i : 1 \leq i \leq l) \in \mathcal{A}_G$. Define the hypergraph $F = F(Q)$ as follows:

$$VF = VG, \quad \mathcal{E}F = (e_i : 1 \leq i \leq l), \quad e_i = VQ_i. \tag{5}$$

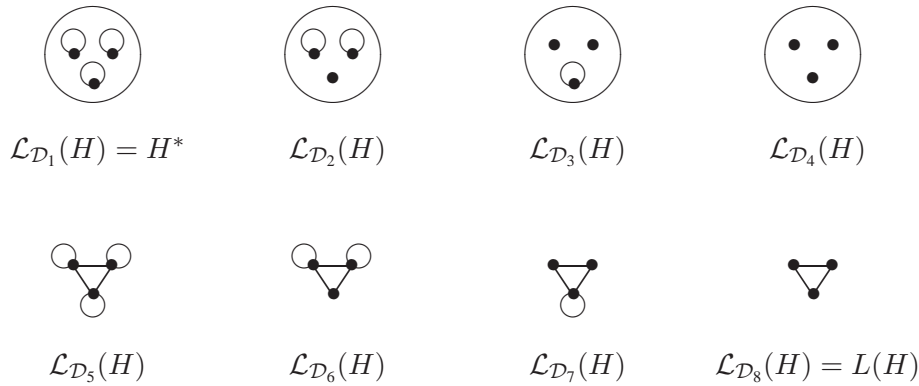


Figure 2. Line hypergraphs of the star $K_{1,3}$.

The hypergraph F does not contain isolated vertices, hence the dual hypergraph F^* exists. The hypergraph F^* is called *canonical (with respect to Q)* and is denoted by $C(Q)$.

If G is a hypergraph without isolated vertices and Q is the covering by edges of G , then $C(Q) = G^*$. Thus, canonical hypergraphs are a generalization of dual hypergraphs.

THEOREM 2 ([30]).

- (1) $\text{rank } Q \in \mathcal{D}_{C(Q)}$ for any $Q \in \mathcal{A}_G$;
- (2) $\mathcal{L}_{\text{rank } Q}(C(Q)) \cong G$;
- (3) If $H \in \mathcal{L}^{-1}(G)$ and $\mathcal{L}_{\mathcal{D}}(H) \cong G$, then there exists $Q \in \mathcal{A}_G$ such that $(H, \mathcal{D}) \cong (C(Q), \text{rank } Q)$.

This theorem is an evolution of Berge’s idea. It is proved in [6] for the case when G is a graph.

COROLLARY 1.

- (1) For any simple hypergraph G , $\mathcal{L}^{-1}(G) = \{C(Q) : Q \in \mathcal{A}_G\}$ and $G \cong \mathcal{L}_{\text{rank } Q}(C(Q))$;
- (2) For $Q_1, Q_2 \in \mathcal{A}_G$, $C(Q_1) \cong C(Q_2)$ if and only if there exists a permutation ψ of VG such that $\psi(P_1) = P_2$, where P_k is a clique covering of VG corresponding to the covering Q_k , $k = 1, 2$.

COROLLARY 2.

- (1) $\mathcal{L}^{-1}(G) \neq \emptyset$ if and only if G is a simple hypergraph;
- (2) The minimal order of hypergraphs from $\mathcal{L}^{-1}(G)$ is equal to $cc(G)$. In particular, if G is a simple triangle-free graph, then the minimal order is equal to the number of edges of G .

Another generalization of line graphs which is not covered by our generalization can be found in [20]

3. Theorems of Krausz and Whitney

The following two classical theorems concerning line graphs are well known. The first is Krausz' theorem giving a global characterization of line graphs.

KRAUSZ' THEOREM ([17]). *A graph G is the line graph of some simple graph if and only if there exists a clique covering Q of G such that*

- (1) *Each vertex of G belongs to exactly two components of Q ;*
- (2) *Each edge of G belongs to exactly one component of Q .*

Analogous characterizations in terms of clique coverings were obtained for line graphs of hypergraphs with prescribed properties, for example, for simple hypergraphs, r -uniform hypergraphs, linear hypergraphs, linear r -uniform hypergraphs and 2-colorable hypergraphs [6, 7, 29].

The second classical theorem is Whitney's theorem on edge isomorphisms.

WHITNEY'S THEOREM ([31]). *If G and H are connected graphs and $L(G) \cong L(H)$, then either $G \cong H$ or $\{G, H\} = \{K_3, K_{1,3}\}$. Further, if the orders of G and H are greater than 4, then for any isomorphism $\alpha: L(G) \rightarrow L(H)$ there exists a unique isomorphism $G \rightarrow H$ inducing α .*

Berge and Rado [8] and Gardner [13] have obtained results for hypergraphs analogous in some sense to this theorem. A relationship for hypergraphs which is stronger than isomorphism has been required for this.

The technique of coverings, namely Corollary 1, has enabled us to unify the theorems of Krausz and Whitney and to state a more general version as a single theorem. Let P be a graph theoretic property. We can envision P as a class of hypergraphs closed with respect to isomorphism. Put

$$P^* = \{H^*: H \in P, H \text{ does not contain isolated vertices}\}.$$

Further, let G be a simple hypergraph and let $F(Q)$ be the hypergraph defined by (5). If $F(Q) \in P$, then we say that Q has property P or $Q \in P$. Obviously, $C(Q) \in P$ if and only if $Q \in P^*$. Thus Corollary 1 implies the following existence and uniqueness theorem.

THEOREM 3.

- (1) $\mathcal{L}^{-1}(G) \cap P \neq \emptyset$ if and only if $\mathcal{A}_G \cap P^* \neq \emptyset$ for any simple hypergraph G and property P ;
- (2) For a simple uniform hypergraph G , $|\mathcal{L}^{-1}(G) \cap P| = 1$ if and only if the group of automorphisms $\text{Aut}(G)$ acts transitively on the set of coverings $\mathcal{A}_G \cap P^*$.

The requirement that a hypergraph be uniform in (2) of Theorem 3 is not essential and is assumed for simplicity.

If, in Theorem 3(1), we take P to be the class of simple graphs, then Krausz' Theorem is obtained. Therefore, all characterizations of line hypergraphs of graphs with prescribed properties which follow directly from Theorem 3(1) are called *Krausz type characterizations* or *P -Krausz characterizations* if P is given. The corresponding clique coverings are called *P -Krausz coverings*.

Theorem 3(2) enables one to obtain analogues of Whitney's Theorem for different classes of hypergraphs.

COROLLARY 3. *Let (P, R) be a pair of properties such that $\text{Aut}(G)$ acts transitively on the set of coverings $\mathcal{A}_G \cap P^*$ for any $G \in R$. Further, let*

$$H_i \in P, \quad G_i \in \mathcal{L}(H_i) \cap R, \quad i = 1, 2 \text{ and } G_1 \cong G_2.$$

Then $H_1 \cong H_2$, and for any isomorphism $\alpha: G_1 \rightarrow G_2$ there exists an isomorphism $H_1 \rightarrow H_2$ inducing α .

Under the conditions of Corollary 3, we say that the *Whitney type theorem* holds for the pair (P, Q) .

Put $\mathcal{L}(P) = \{G : G \in \mathcal{L}(H) \text{ for } H \in P\}$.

Let P^l denote the class of linear hypergraphs without isolated vertices and P_r^l denote the class of linear r -uniform hypergraphs without isolated vertices. In particular, P_2^l is the class of simple graphs. Suppose that P is one of these properties, $G \in \mathcal{L}(P)$, and C is a clique in G . The clique C is called *P -large* if

- (1) $\text{rank } C > 2$ for $P = P^l$,
- (2) $\text{rank } C > 2$ or $|C| > r^2 - r + 1$ for $P = P_r^l$.

LEMMA 1 ([30]). *Any P -large clique of a graph G is a component of every P -Krausz covering of G .*

Lemma 1 and Corollary 3 yield the following statement.

COROLLARY 4 ([30]). *The Whitney type theorem holds for the pair (P, Q) , where (P, Q) is one of the following:*

- (1) $P = P^l$, Q is the class of hypergraphs whose edge degrees are greater than 2;
- (2) $P = P_r^l$, Q is the class of hypergraphs whose maximal cliques of rank ≤ 2 have orders greater than $r^2 - r + 1$;
- (3) P is the class of hypergraphs possessing the Helly property such that the vertex stars are pairwise noncomparable with respect to inclusion, Q is the class of all hypergraphs.

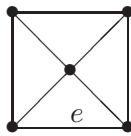


Figure 3. The graph W_4 .

The classical Whitney's Theorem can also be obtained by this scheme. The covering of a hypergraph by cliques with the properties pointed out in Krausz' Theorem is called a *strict linear 2-covering*.

LEMMA 2 ([27]). *Any nontrivial clique of a connected hypergraph $G \in \mathcal{L}_2^l$ is a component of at most one strict linear 2-covering of G .*

This lemma and Corollary 3 together imply the uniqueness of such a covering for all G with the exception of the case when G is a simple K_4 -free graph with maximal degree $\Delta(G) \leq 4$ such that each edge of G belongs to a triangle.

Theorem 4 follows from Lemmas 1 and 2.

THEOREM 4 ([30]). *Let H_1 and H_2 be connected graphs whose line hypergraphs $G_i = \mathcal{L}_{\mathcal{D}_i}(H_i)$ are isomorphic. Then*

- (1) *Either $H_1 \cong H_2$ or $G_i \cong K_3$;*
- (2) *Every isomorphism $G_1 \rightarrow G_2$ is induced by an isomorphism $H_1 \rightarrow H_2$ if and only if G_i is not isomorphic to any of K_3 , $K_4 - e$, $\overline{3K_2}$ or the graph W_4 shown in Figure 3.*

4. Characterizations

Characterizations of some classes of line graphs in terms of forbidden induced subgraphs are obtained on the basis of Krausz type characterizations. There are characterizations for line graphs of the following classes of graphs: simple graphs (Beineke [5]), multigraphs (Bermond and Meyer [10]), multigraphs with restricted multiplicities of edges (Tashkinov [25]), bipartite graphs (Harary and Holzmänn [15]) and bipartite multigraphs (Tyshkevich, Urbanovich and Zverovich [29]).

Zverovich [32] offers a procedure for characterizing line graphs of a strict hereditary class in terms of forbidden induced subgraphs. This procedure is based on Beineke's characterization of line graphs of simple graphs [5] and Whitney's theorem on edge isomorphisms [31].

The problem becomes more complicated when hypergraphs are considered instead of simple graphs. Denote by L_r and L_r^l the classes of line graphs of r -uniform hypergraphs and r -uniform linear hypergraphs, respectively. Lovász [19] stated the problem of characterizing the class L_3 . This class cannot be charac-

terized by a finite list of forbidden induced subgraphs [9]. Levin and Tyshkevich [18] proved that the recognition problem ‘ $G \in L_3$ ’ is *NP*-complete.

Corollary 1 yields the following Krausz type characterization.

COROLLARY 5. *$G \in L_r$ if and only if there exists a clique covering of G such that each vertex of G belongs to at most r components.*

Corollary 5 implies

COROLLARY 6. *For a simple triangle-free graph G the following statements are equivalent:*

- (1) $G \in L_r$;
- (2) Vertex degrees of G do not exceed r ;
- (3) The star $K_{1,r+1}$ is not a subgraph of G .

It is proved in [23] that the class L_3^l cannot be characterized by a finite list of forbidden induced subgraphs. However, line graphs of 3-uniform linear hypergraphs can be characterized by a finite list of forbidden induced subgraphs in the class of graphs whose vertex degrees are at least 69 [23]. In [22], the bound 69 is replaced by 19.

Using Krausz type characterizations and properties of P -large cliques, Tyshkevich, Melnikov and Metelsky [28] worked out an algorithm which recognizes in polynomial time the property ‘ $G \in \mathcal{L}(P)$ ’. If this property is satisfied, then the algorithm constructs one of the hypergraphs $H \in \mathcal{L}^{-1}(G) \cap P$. Here P and G may be taken to be any of the following:

- (1) $P = P_2^l$, G is an arbitrary hypergraph;
- (2) $P = P^l$, G is a hypergraph whose edge degrees are greater than 2;
- (3) $P = P_r^l$, G is such that for any clique C with rank $C > 2$, $|VC| > r^2 - r + 1$.

The representability of a graph as an intersection graph is one of the popular questions in graph theory. Let $F = (S_1, \dots, S_k)$ be a family of pairwise distinct nonempty subsets of a set S . If S is finite, then the *intersection graph* $\Omega(F)$ of the family F is the line graph $L(H)$, where H is a simple hypergraph with $VH = \bigcup_{i=1}^k S_i$ and $EH = F$. Hence the next result follows from Corollary 2.

COROLLARY 7.

- (1) Any simple finite graph is isomorphic to some intersection graph [21];
- (2) If $\omega(G)$ is the minimum number of elements in the set S such that $G \cong \Omega(F)$ for some family F of subsets of S , then

$$\omega(G) \leq m(G) + k(G), \tag{6}$$

where $m(G)$ is the number of edges of G and $k(G)$ is the number of connected components of order ≤ 2 ;

- (3) Inequality (6) is an equality if and only if G is triangle-free.

An important class of intersection graphs is the class of *clique graphs*, i.e., intersection graphs of the families of all maximal cliques of simple graphs. Roberts and Spencer proposed a Krausz type characterization for this class of graphs:

THEOREM 5 ([24]). *A simple graph is a clique graph if and only if it has a clique covering Q such that the hypergraph $F(Q)$ defined by (5) possesses the Helly property.*

An analogous characterization of intersection graphs of k -cliques can be found in [6].

The class of clique graphs is not hereditary. Moreover, any graph can be an induced subgraph of some clique graph.

THEOREM 6 ([12]). *Any simple graph is an induced subgraph of some clique graph. Every triangle-free graph is a clique graph.*

To show this, let H be a simple graph. Put $VG = VH \cup EH$ and

$$G = \bigcup_{v \in VH} G_v$$

where G_v is a complete graph with vertex set $\{v \cup \mathcal{E}_H(v)\}$. Then H is an induced subgraph of the clique graph of G .

5. k -Dimensional Graphs

A graph G is called a *k -dimensional cell graph* if VG is a subset of the Cartesian product $A_1 \times \dots \times A_k$ of finite nonempty sets A_i , and two vertices $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ are adjacent if and only if $x_i = y_i$ for some index i . Any graph isomorphic to a k -dimensional cell graph is called *k -dimensional*. Evidently, every graph is k -dimensional for some k and every k -dimensional graph is $(k + 1)$ -dimensional. The minimal k for which a graph G is k -dimensional is called the *dimension* of G and is denoted by $\dim(G)$.

Babaitsev and Tyshkevich obtained a characterization of k -dimensional graphs.

THEOREM 7 ([3]). *For a simple graph G the following statements are equivalent:*

- (1) $\dim(G) \leq k$;
- (2) $G \cong \mathcal{L}(H)$ where H is a simple k -colorable hypergraph;
- (3) $G \cong \mathcal{L}(H)$ where H is a simple k -uniform k -chromatic hypergraph;
- (4) *There exists a clique covering Q of G satisfying the following two conditions:*
 - (i) Q can be divided into $l \leq k$ parts such that each vertex of G belongs to at most one component of each part,

- (ii) *The intersection of all components having a vertex v is $\{v\}$;*
- (5) *There exists a clique covering Q of G satisfying the following two conditions:*
 - (iii) *Q can be divided into k parts such that each vertex of G belongs to exactly one component of each part,*
 - (iv) *Any k components have at most one vertex in common.*

COROLLARY 8. *The recognition problem ‘ $\dim(G) \leq k$ ’ for fixed $k > 2$ is NP-complete.*

The next few results deal with 2-dimensional graphs.

COROLLARY 9 ([3]). *For a simple graph G the following statements are equivalent:*

- (1) $\dim(G) \leq 2$;
- (2) $G \cong \mathcal{L}(H)$ where H is a simple bipartite graph;
- (3) G does not contain an induced subgraph isomorphic to $K_{1,3}$, $K_4 - e$ or C_{2n+1} , $n \geq 2$.

PROPOSITION 2. *The recognition problem for graph isomorphism is reduced in polynomial time to the analogous problem for two-dimensional graphs.*

This proposition follows directly from Whitney’s Theorem if we make use of the König representation of graphs.

The dimension of a graph is related to the following parameter. Evidently, any graph G can be represented as the union $G = G_1 \cup \dots \cup G_k$ of graphs G_i whose connected components are complete graphs. The minimal number k taken over all such representations of G is called the *equivalent covering number* $\text{eq}(G)$. This number was introduced by Duchet [11] and Behrendt [4].

PROPOSITION 3 ([3]). *For any simple graph G ,*

$$\dim(G) \leq \text{eq}(G) \leq \dim(G) + 1.$$

PROPOSITION 4 ([27]).

- (1) *For any graph G ,*

$$\text{eq}(G) = \min_{Q \in \mathcal{A}_G} \{\chi(\Omega(Q))\} \leq \chi'(G),$$

where χ and χ' are the chromatic number and the chromatic index, respectively, and \mathcal{A}_G is the set of clique coverings;

- (2) *If G is triangle-free, then $\text{eq}(G) = \chi'(G)$.*

COROLLARY 10. *The recognition problem ‘ $\text{eq}(G) \leq k$ ’ for any fixed $k > 2$ is NP-complete.*

The *multiplication* of a vertex v in a graph G by a natural number $l \in \mathbf{N}$ means that we add the graph K_{l-1} and all edges between K_{l-1} and the set $N(v) \cup \{v\}$, where $N(v)$ denotes the neighborhood of v . Denote by $\langle X \rangle$ the closure of a set of graphs X with respect to multiplication of vertices, and let C_k be the set of k -dimensional graphs. Let IG be the set of all independent subsets of the vertex set of G ($\emptyset \in IG$). Then the pair (VG, IG) is called an *independence system*.

THEOREM 8 ([3]). *For a simple graph G the following statements are equivalent:*

- (1) $\text{eq}(G) \leq k$;
- (2) *The independence system (VG, IG) can be represented in the form of an intersection of k matroids;*
- (3) $G \in \langle C_k \rangle$;
- (4) $G = L(H)$ *where H is a k -uniform k -chromatic graph;*
- (5) $G = L(H)$ *where H is a k -colorable graph;*
- (6) *There exists a clique covering of G satisfying (i) of Theorem 7;*
- (7) *There exists a clique covering of G satisfying (iii) of Theorem 7.*

COROLLARY 11 ([27]). *For a simple graph G , the following statements are equivalent:*

- (1) $\text{eq}(G) \leq 2$;
- (2) G *is the line graph of a bipartite multigraph;*
- (3) $G \in \langle C_2 \rangle$;
- (4) G *does not contain an induced subgraph isomorphic to $K_{1,3}$, W_4 , $W_4 - e$ (see Fig. 3) or C_{2n+1} , $n \geq 2$.*

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