Bipartition of Graphs into Subgraphs with Prescribed Hereditary Properties

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Abstract

A hereditary class P is called *finitely generated* if the set of all minimal forbidden induced subgraphs for P is finite. For a pair of hereditary classes P and Q, we define a hereditary class $P * Q$ of all graphs G which have a partition $A \cup B = V(G)$ such that $G(A) \in P$ and $G(B) \in Q$, where $G(X)$ denotes the subgraph of G induced by $X \subseteq V(G)$. We investigate the problem of recognizing finitely generated classes of the form $P * Q$. The following model is used. Let H^0 and H^1 be hypergraphs with the same vertex set V. The ordered pair $H = (H^0, H^1)$ is called a *bihypergraph*. A bihypergraph $H = (H^0, H^1)$ is called *bipartite* if there is an ordered partition $V^0 \cup V^1 = V(H)$ such that V^i is stable in H^i for $i = 0, 1$. If the maximum cardinality of hyperedges in H is at most r and every k-subset of $V(H)$ contains at least one hyperedge, then $H \in C(k, r)$. It was proved in [4] that there exists a finite number of minimal non-bipartite bihypergraphs in $C(k, r)$ whenever k and r are fixed. Let P and Q be hereditary classes of graphs. Suppose that the stability number $\alpha(H)$ is bounded above for all $H \in P$, and the clique number $\omega(H)$ is bounded above for all $H \in Q$. An ordered partition $A \cup B = V(G)$ is called a *Ramseian $P * Q$-partition* if $G(A) \in P$ and $G(B) \in Q$. Let Ramsey$(P * Q)$ be the set of all graphs having a Ramseian $P * Q$-partition. It follows from [4] that if both P and Q are finitely generated, then Ramsey$(P * Q)$ is also finitely generated. In particular, every class of (α, β)-polar graphs generalizing split graphs has a finite forbidden induced subgraph characterization. We formulate a general conjecture that gives conditions for a class of graphs having a (P, Q)-partition to be finitely generated. New results supporting the conjecture are proved.

Keywords: bihypergraph, bipartite hypergraph, hereditary class, (α, β)-polar graph, forbidden induced subgraphs

1 r-Bounded k-Complete Bipartite Bihypergraphs

Let V be a finite set (possibly, empty) and E be a set of subsets of V (possibly, $\emptyset \in E$). The pair $H = (V, E)$ is called a *hypergraph* with the *vertex set* $V = V(H)$ and the *hyperedge set* $E = E(H)$. The number of vertices $n(H) = |V(H)|$ is the *order* of H and the maximum cardinality among hyperedges is the *rank* $r(H)$. A hypergraph H is called *r-bounded* if $r(H) \leq r$. A set $S \subseteq V(H)$ is *stable* in H if S contains no hyperedges of H.

Definition 1 *A set $T \subseteq V(H)$ intersecting every hyperedge of H is called a* transversal. *A* minimal *transversal does not contain any other transversal. A transversal T is called* t-bounded *if $|T| \leq t$. We denote by $MT_t(H)$ the set of all minimal t-bounded transversals in H.*

Note that $\emptyset \in MT_t(H)$ if and only if $E(H) = \emptyset$. Also, H has no transversal if and only if $\emptyset \in E(H)$.

Definition 2 *Let H^0 and H^1 be hypergraphs with the same vertex set V. The ordered pair $H = (H^0, H^1)$ is called a* bihypergraph *with the set of* 0-edges $E(H^0)$ *and the set of* 1-edges $E(H^1)$. *Every hyperedge of either H^0 or H^1 is considered as an* hyperedge *of H.*

The *order* of H is $n(H) = |V|$. The *rank* of H is $r(H) = \max\{r(H^0), r(H^1)\}$.

Definition 3 *For a bihypergraph H, if $r(H) \leq r$ then H is* r-bounded. *For a fixed $k \geq 1$, a bihypergraph H is called* k-complete *if every k-subset of $V(H)$ contains at least one hyperedge. We denote by $C(k, r)$ the class of all k-complete r-bounded bihypergraphs.*

Definition 4 *A bihypergraph $H = (H^0, H^1)$ is called* bipartite *if there exists an ordered partition $V^0 \cup V^1 = V(H)$* (bipartition) *such that the set V^i is stable in H^i, $i = 0, 1$. We denote by $\mathcal{B}$ the set of all bipartite bihypergraphs.*

When $H^0 = H^1$ the problem of recognizing bipartiteness of H coincides with the NP-complete problem of recognizing bipartiteness of H^0. As follows from Theorem 1 below, there is a polynomial algorithm for recognizing bipartite bihypergraphs in the class $C(k, r)$.

Let G be a hypergraph and $X \subseteq V(G)$. By $G(X)$ we denote the *induced subhypergraph* having the vertex set X and the hyperedge set $\{e \in E(G) : e \subseteq X\}$. Note that if $X = \emptyset$, then either $E(G(X)) = \emptyset$ whenever $\emptyset \notin E(G)$ or $E(G(X)) = \{\emptyset\}$ whenever $\emptyset \in E(G)$. Given a bihypergraph $H = (H^0, H^1)$ and a subset X of $V(H)$, the *subhypergraph induced by X* is $H(X) = (H^0(X), H^1(X))$. Let us denote by $\mathrm{ISub}(H)$ the set of all induced subhypergraphs of H.

Definition 5 *A class P of bihypergraphs is* hereditary *if $\mathrm{ISub}(H) \subseteq P$ for every $H \in P$.*

Let Z be a set of bihypergraphs. We put

$$\mathrm{FIS}(Z) = \{H : \mathrm{ISub}(H) \cap Z = \emptyset\}.$$

Clearly, a class P is hereditary if and only if $P = \mathrm{FIS}(Z)$ for a set Z. All bihypergraphs in Z are *forbidden induced subhypergraphs* for $\mathrm{FIS}(Z)$.

Definition 6 *A forbidden induced subhypergraph G for P is a* minimal *if*

$$\mathrm{ISub(G)} \backslash P = \{G\}.$$

Theorem 1 (Zverovich [4]) *Let $k \geq 1$, $r \geq 1$ and $\Psi = \mathrm{FIS}(Z_\Psi) \subseteq C(k,r)$ be a hereditary class of bihypergraphs. Then there is a finite set $Z(\Psi, k, r) \subseteq \Psi$ of bihypergraphs such that*

$$\Psi \cap \mathcal{B} = \mathrm{FIS}(Z_\Psi \cup Z(\Psi, k, r)).$$

Corollary 1 *For fixed k and r, the set of all minimal non-bipartite bihypergraphs in $C(k,r)$ is finite.*

2 Ramseian Partitions of Graphs

The set of all induced subgraphs of a graph G we denote by $\mathrm{ISub}(G)$. A class P of graphs is called *hereditary* if $\mathrm{ISub}(G) \subseteq P$ for every $G \in P$. Let Z be a set of graphs. We put

$$\mathrm{FIS}(Z) = \{G : \mathrm{ISub}(G) \cap Z = \emptyset\}.$$

If $P = \mathrm{FIS}(Z)$, then Z is a set of *forbidden induced subgraphs* for P. A graph H is called a *minimal forbidden induced subgraph* for a hereditary class P if $H \notin P$ and $\mathrm{ISub}(H) \backslash \{H\} \subseteq P$.

Definition 7 *We denote by Z_P the set of all minimal forbidden induced subgraphs for a hereditary class P. We have $P = \mathrm{FIS}(Z_P)$. A hereditary class P is called* finitely generated *if the set Z_P is finite.*

The *stability number* $\alpha(G)$ of a graph G is the maximum cardinality of stable sets in G. The *clique number* $\omega(G)$ of a graph G is the maximum order of complete subgraphs in G. A class P of graphs is called *α-bounded* (respectively, *ω-bounded*) if there is a constant c such that $\alpha(G) \leq c$ (respectively, $\omega(G) \leq c$) for every $G \in P$.

Definition 8 *Let P and Q be classes of graphs. An ordered partition $A \cup B = V(G)$ is called a* (P, Q)-partition *of a graph G if $G(A) \in P$ and $G(B) \in Q$. We denote by $P * Q$ the class of all graphs having a (P, Q)-partition.*

Definition 9 *Let P be an α-bounded hereditary class and Q be an ω-bounded hereditary class. Then a (P, Q)-partition will be called* Ramseian. *We denote by* $\mathrm{Ramsey}(P * Q)$ *the set of all graphs having a Ramseian (P, Q)-partition.*

Theorem 2 (Zverovich [4]) *Let P be an α-bounded hereditary class and Q be an ω-bounded hereditary class. If both P and Q are finitely generated, then the class* $\mathrm{Ramsey}(P * Q)$ *is also finitely generated.*

As an example we consider the classes of *(α, β)-polar* graphs.

The complete graph, the edgeless graph, the path, and the cycle of order n are denoted by K_n, O_n, P_n and C_n, respectively. The union $G_1 \cup G_2 \cup \cdots \cup G_k$ of graphs $G_1, G_2, \ldots, G_k$ is considered as disjoint, i.e., $V(G_i) \cap V(G_j) = \emptyset$ for $1 \leq i \neq j \leq k$. The complementary graph of G is denoted by $\overline{G}$.

Definition 10 (Tyshkevich and Chernyak [3]) *$G \in (\alpha, \beta)$ if and only if there exists a partition $A \cup B = V(G)$ such that*

- *$G(A) \in \mathrm{FIS}(K_1 \cup K_2; O_{\alpha+1})$ (the complete multipartite graphs having the cardinality of parts at most α) and*

- $G(B) \in \mathrm{FIS}(P_3; K_{\beta+1})$ *(disjoint union of complete graphs of orders at most β).*

In particular, the class of all *split* graphs coincides with the class of $(1,1)$-polar graphs and it can be characterized by three forbidden induced subgraphs, namely $2K_2, C_4$ and C_5 (Földes and Hammer [1]). Recently, Gagarin and Metelsky [2] proved that the class of all (1, 2)-polar graphs has a finite FIS-characterization (18 minimal forbidden induced subgraphs). Theorem 2 implies the existence of a finite FIS-characterization for every class of (α, β)-polar graphs whenever α and β are finite. Note that (α, β)-polar graphs are also defined when α is infinite and/or β is infinite [3].

Since each class of (α, β)-polar graphs is a class of type $\mathrm{Ramsey}(P*Q)$, we obtain the following:

Corollary 2 *For every (finite) $\alpha \geq 1$ and $\beta \geq 1$ the class of all (α, β)-polar graphs has a finite forbidden induced subgraph characterization.*

Problem 1 *Let $P = \mathrm{FIS}(O_2)$ (complete graphs) and $Q = \mathrm{FIS}(K_3)$ (triangle-free graphs). Find a forbidden induced subgraph characterization for the class* $\mathrm{Ramsey}(P * Q)$.

3 Families of Hereditary Classes of Graphs

Our aim is to partially prove the following general conjecture.

Conjecture 1 *Let P and Q be hereditary classes distinct from the class of all graphs. The class of all graphs having a (P, Q)-partition is finitely generated if and only if either*
(i) *P is an α-bounded finitely generated class and Q is an ω-bounded finitely generated class (or vice versa) or*
(ii) *P is a finite class and Q is a finitely generated class (or vice versa).*

Definition 11 *We define eight families of hereditary classes $J(\alpha, \omega)$, where $J \in \{F, I\}$ and $\alpha, \omega \in \{c, \infty\}$ in the following way:*

- *if $J = F$ (respectively, $J = I$) then every class in $J(\alpha, \omega)$ is (respectively, is not) finitely generated;*
- *if $\alpha = c$ (respectively, $\alpha = \infty$) then every class in $J(\alpha, \omega)$ is (respectively, is not) α-bounded;*
- *if $\omega = c$ (respectively, $\omega = \infty$) then every class in $J(\alpha, \omega)$ is (respectively, is not) ω-bounded.*

It is supposed that every family $J(\alpha, \omega)$ contains all hereditary classes satisfying these conditions.

Thus, we partition all hereditary classes into eight families:

- $F(c, c)$;
- $F(c, \infty)$;
- $F(\infty, c)$;
- $F(\infty, \infty)$;
- $I(c, c)$;
- $I(c, \infty)$;

- $I(\infty, c)$;
- $I(\infty, \infty)$.

Proposition 1 $I(c, c) = \emptyset$.

Proof. By Ramsey's theorem, every class $P \in I(c, c)$ is finite. Let us denote

$$N = \max\{|V(G)| : G \in P\}.$$

We put $Z = Z_1 \cup Z_2$, where

$$Z_1 = \{H : |V(H)| \leq N \text{ and } H \notin P\}$$

and Z_2 is the set of all graphs of the order $N + 1$. Clearly, the set Z is finite and $Z_P \subseteq Z$. It contradicts to the fact that $P \in I(c, c)$ is not finitely generated. ∎

We will use the following hereditary classes:

- $O = \text{FIS}(K_2)$ is the class of all edgeless graphs;
- $K = \text{FIS}(O_2)$ is the class of all complete graphs;
- $O \cup K = \text{FIS}(P_3, \overline{P}_3)$ is the union of O and K;
- $M = \text{FIS}(\overline{P}_3)$ is the class of all complete multipartite graphs;
- $M_\alpha = \text{FIS}(\overline{P}_3, O_{\alpha+1})$ is the class of all complete multipartite graphs with cardinalities of parts at most α ($\alpha \geq 1$);
- $D = \text{FIS}(P_3)$ is the class of all graphs which are disjoint union of complete graphs;
- $D_\beta = \text{FIS}(P_3, K_{\beta+1})$ is the class of all graphs which are disjoint union of complete graphs of order at most β ($\beta \geq 1$).

Now we reformulate Theorem 2 as

Theorem 3 *For every hereditary classes $P \in F(c, \omega)$ and $Q \in F(\alpha, c)$, where $\alpha, \omega \in \{c, \infty\}$, the class $P * Q$ is finitely generated.*

This result leads to the following problem.

Problem 2 *Recognize pairs $\{J(\alpha, \omega), J'(\alpha', \omega')\}$ such that for every hereditary classes $P \in J(\alpha, \omega)$ and $Q \in J'(\alpha', \omega')$ the class $P * Q$ is finitely generated.*

A pair $\{J(\alpha, \omega), J'(\alpha', \omega')\}$ is considered as unordered since $P * Q = Q * P$. Theorem 3 gives a positive solution for pairs

$$\{F(c, c), F(c, c)\}, \{F(c, c), F(\infty, c)\}, \{F(c, c), F(c, \infty)\} \text{ and } \{F(c, \infty), F(\infty, c)\}.$$

We will show that the positive solution of Problem 2 there exists also for pair $\{F(c, c), F(\infty, \infty)\}$. Let us consider the pair $\{F(c, c), F(\alpha, \omega)\}$.

Definition 12

- *A hypergraph H is called* hereditary *if $\emptyset \in E(H)$ and $2^e \subseteq E(H)$ for each edge $e \in E(H)$.*

- *A hypergraph H is called* r-bounded ($r \geq 0$) *if $|e| \leq r$ for each $e \in E(H)$.*

We denote by $H(r,s)$, where $r \geq 0$ and $s \geq 0$, the set of all bihypergraphs $H = (H^0, H^1)$ in which H^0 is a hereditary r-bounded hypergraph while H^1 is an s-bounded hypergraph and $\emptyset \notin E(H^1)$.

Definition 13 *A 0-edge e is called a* 0-transversal *of a bihypergraph if $e \cap f \neq \emptyset$ for each 1-edge f. A bihypergraph $H \in H(r,s)$ is called* 0-minimal *if there are no 0-transversals in it, but every induced subbihypergraph $H(X) \neq H$ has a 0-transversal.*

Theorem 4 *For any fixed $r \geq 0$ and $s \geq 0$, the set of all 0-minimal bihypergraphs in $H(r,s)$ is finite.*

Proof. Let $H = (H^0, H^1) \in H(r,s)$ and H be a 0-minimal bihypergraph. If $E^1 = E(H^1) = \emptyset$, then $\emptyset \in E^0 = E(H^0)$ is a 0-transversal in H, contrary to the definition of 0-minimal bihypergraphs. Hence $E^1 \neq \emptyset$.

We define a sequence $B_0, B_1, \ldots, B_r$ of induced subbihypergraphs of H as follows.
Step 0. Choose an arbitrary 1-edge e in H and put $V_0 = e$ and $B_0 = H(V_0)$.
Step i + 1 ($0 \leq i < r$). Let the bihypergraph B_i be already defined. For each 0-edge f in H, we assign a 1-edge f^* which does not intersect f. The existence of f^* follows from the fact that none of 0-edges is a 0-transversal. We put $W_i = \cup f^*$, where the union is taken over all 0-transversals f of the bihypergraph B_i. Further, $V_{i+1} = V_i \cup W_i$ and $B_{i+1} = H(V_{i+1})$.

Let $n_i = |V_i| = |V(B_i)|$. By the construction, $n_{i+1} < n_i + s2^{n_i}$ ($i = 0, 1, \ldots, r-1$), since the number of 0-transversal in B_i is less than 2^{n_i} and $|f^*| \leq s$ for each 1-edge f^*. It follows from $n_0 \leq r$ that n_r is bounded above by a function $f(r,s)$.

We show that the bihypergraph B_r has no 0-transversal. Let e be an arbitrary 0-transversal in B_r. Denote $e_i = e \cap V_i$ for $i = 0, 1, \ldots, r$. By hereditariness, $e_i \in E^0$ and so e_i is a 0-edge of B_i. We use induction on i to prove that $|e_i| \geq i + 1$, $i = 0, 1, \ldots, r$. The basis of induction: since $V_0 = V(B_0)$ is a 1-edge and e is a 0-transversal, $e \cap V_0 \neq \emptyset$, i.e., $|e_0| \geq 1$. Now suppose that $|e_i| \geq i + 1$, where $i \in \{0, 1, \ldots, r-1\}$ and show that

$$|e_{i+1}| \geq i + 2.$$

Since e is a 0-transversal in B_r and B_i is a subbihypergraph of B_r, e_i is a 0-transversal in B_i. By the definition, B_{i+1} contains a 1-edge e_i^* not intersecting e_i. Since e_i is a 0-transversal in B_r, $e \cap e_i^* \neq \emptyset$. Therefore $|e_{i+1}| \geq |e_i| + |e \cap e_i| \geq i + 2$. In particular, $|e_r| = |e| \geq r + 1$, a contradiction to the condition that H^0 is r-bounded. This contradiction shows that H_r has no transversal. By 0-minimality, H is a subbihypergraph of H_r. Hence $|V(H)| \leq f(r,s)$. ∎

Corollary 3 *If a hereditary class P is finite and a hereditary class Q is finitely generated, then the class $P * Q$ is also finitely generated.*

Proof. Let $Z_Q = \{G_1, G_2, \ldots, G_k\}$. We assign a bihypergraph $H = (H^0, H^1)$ to every graph G: H has the vertex set $V = V(G)$, the set of 0-edges

$$E^0 = \{X \subseteq V : X \neq \emptyset \text{ and } G(X) \in P\} \cup \{\emptyset\}$$

and the set of 1-edges

$$E^1 = \{X \subseteq V : X \neq \emptyset \text{ and } G(X) \in Z_Q\}.$$

Let $r = \max\{|V(G)| : G \in P\}$. Since P is hereditary and finite, H^0 is hereditary and r-bounded. The set Z_Q is finite, and so H^1 is s-bounded, where $s = \max\{|V(G)| : G \in Z_Q\}$.

Thus, $H \in H(r,s)$ for any graph G. If G is a forbidden induced subgraph for $P * Q$, then H has no 0-transversal. If G is a minimal forbidden induced subgraph for $P * Q$, then H is 0-minimal. By Theorem 4, the number of 0-minimal bihypergraphs in $H(r,s)$ is finite. Moreover, there are finitely many minimal forbidden induced subgraphs for $P * Q$ with the same assigned bihypergraph. ∎

Corollary 4 *If $P \in F(c,c)$ and $Q \in F(\alpha,\omega)$, where $\alpha,\omega \in \{c,\infty\}$, then the class $P*Q$ is finitely generated.*

Proof. By Ramsey's theorem, a class $P \in F(c,c)$ is finite and the result follows from Corollary 3. ∎

4 Constructions of Infinite Series of Minimal Forbidden Induced Subgraphs

We will show that Problem 2 for a pair $\{J(\alpha,\omega), J'(\alpha',\omega')\}$ has a negative solution except cases covered by Theorem 3 and Corollary 4. In other words, we show that there exist hereditary classes $P \in J(\alpha,\omega)$ and $Q \in J'(\alpha',\omega')$ such that the class $P * Q$ is not finitely generated.

The number of cases may be reduced by excluding the *complementary* class $\overline{P} = \{\overline{G} : G \in P\}$. A class $P*Q$ is finitely generated if and only if the class $\overline{P * Q} = \overline{P} * \overline{Q}$ is finitely generated. Clearly that if $P \in J(\alpha,\omega)$ then $\overline{P} \in J(\omega,\alpha)$. So we will only consider one of two pairs $\{J(\alpha,\omega), J'(\alpha',\omega')\}$ and $\{J(\omega,\alpha), J'(\omega',\alpha')\}$.

Case 1: $\{F(\infty,c), F(\infty,c)\}$ (the complementary case is $\{F(c,\infty), F(c,\infty)\}$)

We choose the class $O = \mathrm{FIS}(K_2) \in F(\infty,c)$. By König's theorem, the class $O * O$ of all bipartite graphs is $B = \mathrm{FIS}(C_{2n+1} : n \geq 1)$ and it is not finitely generated.

Case 2: $\{F(c,\infty), F(\infty,\infty)\}$ (the complementary case is $\{F(\infty,c), F(\infty,\infty)\}$)

We choose classes $K = \mathrm{FIS}(O_2) \in F(c,\infty)$ and $D = \mathrm{FIS}(P_3) \in F(\infty,\infty)$.

Proposition 2 *The class $K * D$ of all $(1,\infty)$-polar graphs is not finitely generated.*

Proof. For any $n \geq 2$, we prove that the graph $G = \overline{C}_{2n+1}$ is a minimal forbidden induced subgraph for $K*D$. Suppose that $G \in K*D$, i.e., there exists a partition $A \cup B = V(G)$ such that $G(A) \in K$ and $G(B) \in D$. Let $(v_0, v_1, \ldots, v_{2n})$ be the cyclic ordering of vertices of $\overline{G} = C_{2n+1}$. We will refer to vertices v_i and v_{i+1} $(i = 0,1,\ldots,2n-1)$ as well as vertices v_{2n} and v_0 as *neighbors.* Since A induces a complete subgraph and neighbors are non-adjacent, A contains at most one vertex of each pair of neighbors. It follows that there are neighbors in B. Taking symmetry into account, we may assume that $v_0, v_1 \in B$. Since $\{v_0, v_1, v_3\}$ induces P_3, $v_3 \notin B$. So $v_3 \in A$. Then $v_2, v_4 \in B$. We see that $\{v_1, v_2, v_4\} \subseteq B$ induces P_3, a contradiction to $G(B) \in D = \mathrm{FIS}(P_3)$. Thus, $G \notin K * D$. Now we prove the minimality of G. Each induced subgraph $H \neq G$ is the complement of a bipartite graph. Hence there is a partition $A \cup B = V(H)$ such that both A and B induce complete subgraphs. It follows from $K \subseteq D$ that $H \in K * D$. ∎

Case 3: $\{F(\infty,\infty), F(\infty,\infty)\}$

We consider the class $O \cup K = \mathrm{FIS}(P_3, \overline{P}_3) \in F(\infty,\infty)$.

Proposition 3 *The class $(O \cup K) * (O \cup K)$ is not finitely generated.*

Proof. We show that $G = C_{2n+1}$, $n \geq 2$, is a minimal forbidden induced subgraph for $(O \cup K) * (O \cup K)$.

Suppose that $G \in (O \cup K) * (O \cup K)$, i.e., there exists a partition $A \cup B = V(G)$ such that $G(A), G(B) \in (O \cup K) * (O \cup K)$.

Let $(v_0, v_1, \ldots, v_{2n})$ be the cyclic ordering of vertices in G. Since the number of all vertices is odd, there are two neighbors in the same class. By the symmetry, we may assume that $v_0, v_1 \in A$. Then A induces a complete subgraph. Hence $v_2, v_3, v_4 \in B$. The set $\{v_2, v_3, v_4\}$ induces P_3, a contradiction to $G(B) \in \mathrm{FIS}(P_3, \overline{P}_3)$. $G \notin (O \cup K) * (O \cup K)$.

Now we prove minimality of G. Any induced subgraph $H \neq G$ is bipartite. So there is a partition $A \cup B = V(H)$ such that both A and B induce edgeless subgraphs. Since $O \subseteq O \cup K$, $H \in (O \cup K) * (O \cup K)$. ∎

Case 4: $\{F(c, c), I(\alpha, \omega)\}$, where $\alpha, \omega \in \{c, \infty\}$

We consider the following classes:

- $P = \mathrm{FIS}(K_1) = \emptyset \in F(c, c)$ and
- $Q_1 = \mathrm{FIS}(C_n : n \geq 3) \in I(\infty, c)$ (for $\alpha = \infty$, $\omega = c$);
- $Q_2 = \mathrm{FIS}(\overline{C}_n : n \geq 3) \in I(c, \infty)$ (for $\alpha = c$, $\omega = \infty$);
- $Q_3 = \mathrm{FIS}(C_n : n \geq 4) \in I(\infty, \infty)$ (for $\alpha = \omega = \infty$)

(by Proposition 1, $I(c, c) = \emptyset$).

Clearly that for each Q_i above, the class $P * Q_i = Q_i$ is not finitely generated.

Case 5: $\{F(\infty, c), I(\infty, c)\}$ (the complementary case is $\{F(c, \infty), I(c, \infty)\}$)

We consider the class $O = \mathrm{FIS}(K_2) \in F(\infty, c)$ of all edgeless graphs and $B = \mathrm{FIS}(C_{2n+1} : n \geq 1) \in I(\infty, c)$ of all bipartite graphs.

Proposition 4 *The class $O * B$ of all 3-colorable graphs is not finitely generated.*

Proof. For any $n \geq 1$, we construct a graph G_n as follows. Let C_{2n+1} be a cycle with the vertex set $(v_0, v_1, \ldots, v_{2n})$ (in the cyclic order). We add new vertices u_{2i-1} $(i = 1, 2, \ldots, n)$ along with edges $v_{2i-2}u_{2i-1}$, $v_{2i-1}u_{2i-1}$ and $v_{2i}u_{2i-1}$.

It is not hard to check that G_n is a minimal forbidden induced subgraph for $O * B$. ∎

Case 6: $\{F(c, \infty), I(\infty, c)\}$ (the complementary case is $\{F(\infty, c), I(c, \infty)\}$)

We choose the classes $K = \mathrm{FIS}(O_2) \in F(c, \infty)$ of all complete graphs and $AC = \mathrm{FIS}(C_n : n \geq 3) \in I(\infty, c)$ of all acyclic graphs.

Proposition 5 *The class $K * AC$ is not finitely generated.*

Proof. We show that, for all $m \geq 3$ and $n \geq 3$, the graph $G = C_m \cup C_n$ is a minimal forbidden induced subgraph for the class $K * AC$.

Suppose that $G \in K * AC$, i.e., there exists a partition $A \cup B = V(G)$ such that $G(A) \in K$ and $G(B) \in AC$. Since A induces a complete graph then either $A \cap V(C_m) = \emptyset$ or $A \cap V(C_n) = \emptyset$. In any case, $G(B)$ contains a cycle, a contradiction. Thus, $G \notin K * AC$.

Let us prove the minimality of G. Each induced subgraph $H \neq G$ contains at most one cycle. Therefore there is a vertex $u \in V(H)$ such that $H - u$ has no cycles. We put $A = \{u\}$ and $B = V(H) \backslash A$. ∎

Case 7: $\{F(c,\infty), I(\infty,\infty)\}$ the complementary case is $\{F(\infty,c), I(\infty,\infty)\}$)

We consider the classes $K = \text{FIS}(O_2) \in F(c,\infty)$ of all complete graphs and $T = \text{FIS}(C_n : n \geq 4) \in I(\infty,\infty)$ of all triangulated graphs.

The class $K * T$ is not finitely generated because there exists an infinite series $\{C_m \cup C_n : m,n \geq 4\}$ of minimal forbidden induced subgraphs. The proof is essentially the same as in Proposition 5.

Case 8: $\{F(\infty,\infty), I(\infty,c)\}$ (the complementary case is $\{F(\infty,\infty), I(c,\infty)\}$)

We consider the classes $O \cup K = \text{FIS}(P_3, \overline{P}_3) \in F(\infty,\infty)$ of all edgeless and complete graphs and $AC = \text{FIS}(C_n : n \geq 3) \in I(\infty,c)$.

Proposition 6 *The class $(O \cup K) * A$ is not finitely generated.*

Proof. We show that $G = K_4 \cup C_n$ $(n \geq 3)$ is a minimal forbidden induced subgraph for $(O \cup K) * AC$.

Suppose that $G \in (O \cup K) * AC$, i.e., there is a partition $A \cup B = V(G)$ such that $G(A) \in O \cup K$ and $G(B) \in AC$. Since $G(B)$ has no cycles, at least two vertices of K_4 belong to A. Then A induces a complete subgraph. So C_n is a cycle of $G(B)$, a contradiction to $G(B) \in AC$.

Now we prove the minimality of G. Any induced subgraph $H \neq G$ either does not contain C_n (in that case we put $A = V(H) \cap V(K_4)$ and $B = V(H) \backslash A$), or does not contain K_4 as a component (in that case we put $A = \{u, v\}$, where $u \in V(C_n) \cap V(H)$, $v \in V(K_4) \cap V(H)$ and $B = V(H) \backslash A$). ∎

Case 9: $\{F(\infty,\infty), I(\infty,\infty)\}$

We choose the classes $O \cup K = \text{FIS}(P_3, \overline{P}_3) \in F(\infty,\infty)$ of all edgeless and complete graphs and $T = \text{FIS}(C_n : n \geq 4) \in I(\infty,\infty)$ of all triangulated graphs.

Proposition 7 *The class $(O \cup K) * T$ is not finitely generated.*

Proof. Define a graph G which consists of a cycle $C = C_n$ with vertex set $(v_1, v_2, \ldots, v_n)$ (vertices are ordered cyclically) and disjoint cycles $H_i = C_4$, $i = 1, 2, \ldots, n$, with the condition that every vertex v_i is adjacent to every vertex of $V(H_i)$.

We show that, for all $n \geq 4$, G is a minimal forbidden induced subgraph for the class $(O \cup K) * T$.

Suppose that $G \in (O \cup K) * T$, i.e., there exists a partition $A \cup B = V(G)$ such that $G(A) \in O \cup K$ and $G(B) \in T$. Since $G(B)$ has no C_4's, at least one vertex of each $H_i = C_4$ belongs to A. So $G(A)$ is an edgeless graph. Then C is contained in $G(B)$, a contradiction to $G(B) \in T$. Thus, $G \notin (O \cup K) * T$.

Now we prove the minimality of G. It is sufficient to show that $G - v \in (O \cup K) * T$ for every vertex $v \in V(G)$.

If $v \in V(H_i)$ then we include into A a vertex of each subgraph H_j, $j = 1, 2, \ldots, i-1, i+1, \ldots, n$, and v_i. Clearly, $G(A) \in O \subset O \cup K$. The set $B = V(G - v) \backslash A$ induces a triangulated graph. So $G - v \in (O \cup K) * T$.

If $v \in V(C)$ then we include into a vertex of each subgraph H_j, $j = 1, 2, \ldots, n$. It is clear that A induces an edgeless graph while $B = V(G - v) \backslash A$ induces a triangulated graph. ∎

Case 10: $\{I(\infty,c), I(\infty,c)\}$ (the complementary case is $\{I(c,\infty), I(c,\infty)\}$)

We consider the class $B = \text{FIS}(C_{2n+1} : n \geq 1) \in I(\infty,c)$ of all bipartite graphs.

Proposition 8 *The class $B * B$ is not finitely generated.*

Proof. For any $n \geq 1$, we construct a graph G_n in the following way. Let C_{2n+1} be a cycle with the vertex set $(v_0, v_1, \ldots, v_{2n})$ (vertices are ordered cyclically). Add vertices u_{2i-1} and w_{2i-1} $(i = 1, 2, \ldots, n)$ along with the edges $v_{2i-2}u_{2i-1}$, $v_{2i-2}w_{2i-1}$, $v_{2i-1}u_{2i-1}$, $v_{2i-1}w_{2i-1}$, $v_{2i}u_{2i-1}$, $v_{2i}w_{2i-1}$ and $u_{2i-1}w_{2i-1}$.

It is not hard to check that G_n is a minimal forbidden induced subgraph for $B * B$. ∎

Case 11: $\{I(c,\infty), I(\infty,c)\}$

We consider the classes $\overline{AC} = \text{FIS}(\overline{C}_n : n \geq 3) \in I(c,\infty)$ (complements to acyclic graphs) and $AC = \text{FIS}(C_n : n \geq 3) \in I(\infty,c)$ (acyclic graphs).

Proposition 9 *The class $\overline{AC} * AC$ is not finitely generated.*

Proof. We show that the graph $G = C_l \cup C_m \cup C_n$ $(l, m, n \geq 3)$ is a minimal forbidden induced subgraph for the class $\overline{AC} * AC$.

Suppose that $G \in \overline{AC} * AC$, i.e., there is a partition $A \cup B = V(G)$ such that $G(A) \in \overline{AC}$ and $G(B) \in AC$. Since $G(B)$ has no cycles, at least one vertex of each of cycles C_l, C_m and C_n belongs to A. These three vertices induce $\overline{C}_3$, a contradiction to $G(A) \in \overline{AC}$. Thus, $G \notin \overline{AC} * AC$.

Now we prove the minimality of G. Every induced subgraph $H \neq G$ contains at most two cycles. Let us include into A a vertex of every cycle of H. Clearly, $G(A) \in \overline{AC}$ and the set $B = V(H) \backslash A$ induces an acyclic graph. ∎

Case 12: $\{I(\infty,\omega), I(\infty,\infty)\}$, $\omega \in \{c,\infty\}$ (the complementary case is $\{I(\omega,\infty), I(\infty,\infty)\}$)

We consider classes $AC = \text{FIS}(C_n : n \geq 3) \in I(\infty,c)$ of all acyclic graphs and $T = \text{FIS}(C_n : n \geq 4) \in I(\infty,\infty)$ of all triangulated graphs.

Proposition 10 *The classes $AC * T$ and $T * T$ are not finitely generated.*

Proof. We define a graph G of order $6n+3$ with the vertex set $V_0 \cup V_1 \cup \cdots \cup V_{2n}$, where $|V_i| = 3$ for all $i = 0, 1, \ldots, 2n$. Each vertex of V_i is adjacent to each vertex of V_{i+1} $(i = 0, 1, \ldots, 2n)$ and each vertex of V_{2n} is adjacent to each vertex of V_0. We show that, for every $n \geq 1$, G is a minimal forbidden induced subgraph for $AC * T$ and $T * T$.

Suppose that $G \in T * T$, i.e., there exists a partition $A \cup B = V(G)$ such that $G(A) \in T$ and $G(B) \in T$. If $|V_i \cap A| \geq 2$ then V_i is called an *A-set*. Otherwise V_i is a *B-set*. Since the number of sets V_i is odd, there are either two neighboring A-sets or two neighboring B-sets. Taking symmetry into account, we may assume that $|V_0 \cap A| \geq 2$ and $|V_1 \cap A| \geq 2$. We choose vertices $u_i, v_i \in V_i \cap A$, $i = 0, 1$. It is clear that $\{u_0, u_1, v_0, v_1\}$ induces C_4, a contradiction to $G(A) \in T$. Thus, $G \notin T * T$. Since $AC * T \subset T * T$, $G \notin AC * T$.

Now we prove the minimality of G. It is sufficient to show that $G - v \in AC * T$ for each vertex $v \in V(G)$. Since $AC * T \subset T * T$, it will follow that $G - v \in T * T$. Taking symmetry into account, we may assume that $v \in V_0$. Let $V_0 \backslash \{v\} = \{x, y\}$. We put

$$A = \{x\} \cup (\bigcup_{i=1}^{n} V_{2i-1})$$

and

$$B = \{y\} \cup (\bigcup_{i=1}^{n} V_{2i}).$$

Clearly, $G(A) \in AC$ and $G(B) \in AC \subset T$. ∎

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