

A generalised upper bound for the k -tuple domination number

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Abstract

In this paper, we provide an upper bound for the k -tuple domination number that generalises known upper bounds for the double and triple domination numbers. We prove that for any graph G ,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\sum_{m=1}^{k-1} (k - m)\widehat{d}_m + \varepsilon) + 1}{\delta - k + 2} n,$$

where $\gamma_{\times k}(G)$ is the k -tuple domination number; δ is the minimal degree; \widehat{d}_m is the m -degree of G ; $\varepsilon = 1$ if $k = 1$ or 2 and $\varepsilon = -d$ if $k \geq 3$; d is the average degree.

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We consider finite and undirected graphs without loops and multiple edges. If G is a graph, then $V(G) = \{v_1, v_2, \dots, v_n\}$ is the set of vertices in G , d_i denotes the degree of v_i and $d = (\sum_{i=1}^n d_i)/n$ is the average degree of G . Let $N(x)$ denote the neighbourhood of a vertex x . Also let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$, where X is a vertex set of G . Denote by $\delta(G)$ the minimal vertex degree of G , and put $\delta = \delta(G)$.

A set $X \subseteq V(G)$ is called a *dominating set* if every vertex not in X is adjacent to a vertex in X . The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$. A set X is called a *k -tuple dominating set* of G if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. The minimum cardinality of a k -tuple dominating set of G is the *k -tuple domination number* $\gamma_{\times k}(G)$. It is easy to see that the k -tuple domination number is only defined for graphs with $\delta \geq k - 1$. Also, $\gamma(G) = \gamma_{\times 1}(G)$ and $\gamma_{\times k}(G) \leq \gamma_{\times k'}(G)$ for $k \leq k'$. The 2-tuple domination number $\gamma_{\times 2}(G)$ and the 3-tuple domination number $\gamma_{\times 3}(G)$ are called the *double domination number* and the *triple domination number*, respectively. A number of interesting results on the k -tuple domination number can be found in [3–9,11].

Alon and Spencer [1], Arnaoutov [2] and Payan [10] independently proved the following fundamental result:

Theorem 1 (Alon and Spencer [1], Arnaoutov [2], Payan [10]). *For any graph G ,*

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n.$$

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Harant and Henning [3] found an upper bound for the double domination number:

Theorem 2 (Harant and Henning [3]). For any graph G with $\delta \geq 1$,

$$\gamma_{\times 2}(G) \leq \frac{\ln \delta + \ln(d + 1) + 1}{\delta} n.$$

An interesting upper bound for the triple domination number was given by Rautenbach and Volkmann [11]:

Theorem 3 (Rautenbach and Volkmann [11]). For any graph G with $\delta \geq 2$,

$$\gamma_{\times 3}(G) \leq \frac{\ln(\delta - 1) + \ln(d + \widehat{d}_2) + 1}{\delta - 1} n,$$

where $\widehat{d}_2 = (1/n) \sum_{i=1}^n \binom{d_i}{2}$.

The following theorem generalises this bound for the k -tuple domination number. For $m \leq \delta$, let us define the m -degree \widehat{d}_m of a graph G as follows:

$$\widehat{d}_m = \widehat{d}_m(G) = \frac{1}{n} \sum_{i=1}^n \binom{d_i}{m}.$$

Note that \widehat{d}_1 is the average degree d of a graph and $\widehat{d}_0 = 1$.

Theorem 4. For any graph G with $3 \leq k \leq \delta + 1$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln((k - 2)d + (2k - 5)\widehat{d}_2 + \sum_{m=3}^{k-1} (k - m)\widehat{d}_m) + 1}{\delta - k + 2} n.$$

Proof. Let A be formed by an independent choice of vertices of G , where each vertex is selected with probability p , $0 \leq p \leq 1$. For $m = 0, 1, \dots, k - 1$, let us denote

$$B_m = \{v_i \in V(G) - A : |N(v_i) \cap A| = m\}.$$

Also, for $m = 0, 1, \dots, k - 2$, we denote

$$A_m = \{v_i \in A : |N(v_i) \cap A| = m\}.$$

For each set A_m , we form a set A'_m in the following way. For every vertex in the set A_m , we take $k - m - 1$ neighbours not in A . Such neighbours always exist because $\delta \geq k - 1$. It is obvious that $|A'_m| \leq (k - m - 1)|A_m|$. For each set B_m , we form a set B'_m by taking $k - m - 1$ neighbours not in A for every vertex in B_m . We have $|B'_m| \leq (k - m - 1)|B_m|$.

We construct the set D as follows:

$$D = A \cup \left(\bigcup_{m=0}^{k-2} A'_m \right) \cup \left(\bigcup_{m=0}^{k-1} B_m \cup B'_m \right).$$

The set D is a k -tuple dominating set. Indeed, if there is a vertex v which is not k -tuple dominated by D , then v is not k -tuple dominated by A . Therefore, v would belong to A_m or B_m for some m , but all such vertices are k -tuple dominated by the set D by construction.

The expectation of $|D|$ is

$$\begin{aligned} E(|D|) &\leq E\left(|A| + \sum_{m=0}^{k-2} |A'_m| + \sum_{m=0}^{k-1} |B_m| + \sum_{m=0}^{k-1} |B'_m|\right) \\ &\leq E\left(|A| + \sum_{m=0}^{k-2} (k-m-1)|A_m| + \sum_{m=0}^{k-1} (k-m)|B_m|\right) \\ &= E(|A|) + \sum_{m=0}^{k-2} (k-m-1)E(|A_m|) + \sum_{m=0}^{k-1} (k-m)E(|B_m|). \end{aligned}$$

We have

$$E(|A|) = \sum_{i=1}^n P(v_i \in A) = pn.$$

Also,

$$\begin{aligned} E(|A_m|) &= \sum_{i=1}^n P(v_i \in A_m) = \sum_{i=1}^n p \binom{d_i}{m} p^m (1-p)^{d_i-m} \\ &\leq p^{m+1} (1-p)^{\delta-m} \sum_{i=1}^n \binom{d_i}{m} = p^{m+1} (1-p)^{\delta-m} \widehat{d}_m n \end{aligned}$$

and

$$\begin{aligned} E(|B_m|) &= \sum_{i=1}^n P(v_i \in B_m) = \sum_{i=1}^n (1-p) \binom{d_i}{m} p^m (1-p)^{d_i-m} \\ &\leq p^m (1-p)^{\delta-m+1} \sum_{i=1}^n \binom{d_i}{m} = p^m (1-p)^{\delta-m+1} \widehat{d}_m n. \end{aligned}$$

Therefore,

$$\begin{aligned} E(|D|) &\leq pn + (k-1)E(|A_0|) + (k-2)E(|A_1|) + (k-3)E(|A_2|) + \sum_{m=3}^{k-2} (k-m-1)E(|A_m|) \\ &\quad + kE(|B_0|) + (k-1)E(|B_1|) + (k-2)E(|B_2|) + \sum_{m=3}^{k-1} (k-m)E(|B_m|). \end{aligned}$$

Let us denote

$$\mu = \delta - k + 2.$$

Since $k \geq 3$, we have

$$(1-p)^{\delta-1} \leq (1-p)^{\delta-k+2} = (1-p)^\mu.$$

Using the inequality $1-x \leq e^{-x}$, we obtain

$$(1-p)^{\delta-1} \leq e^{-p\mu}.$$

For the second and third terms of the above bound for $E(|D|)$, we have

$$\begin{aligned} (k-1)E(|A_0|) + (k-2)E(|A_1|) &\leq (k-1)p(1-p)^\delta n + (k-2)p^2(1-p)^{\delta-1} \widehat{d}_1 n \\ &\leq (k-1)p(1-p)e^{-p\mu} n + (k-2)p^2 de^{-p\mu} n. \end{aligned}$$

Let us consider the fourth term $(k - 3)E(|A_2|)$. We may assume that $k \geq 4$, for otherwise $k = 3$ and all the inequalities in (1) are true. Note that for $k \geq 4$,

$$(1 - p)^{\delta-2} \leq (1 - p)^{\delta-k+2} = (1 - p)^\mu \leq e^{-p\mu}.$$

We obtain

$$(k - 3)E(|A_2|) \leq (k - 3)p^3(1 - p)^{\delta-2}\widehat{d}_2n \leq (k - 3)p^3\widehat{d}_2e^{-p\mu}n. \tag{1}$$

Furthermore,

$$\begin{aligned} \sum_{m=3}^{k-2} (k - m - 1)E(|A_m|) &\leq \sum_{m=3}^{k-2} (k - m - 1)p^{m+1}(1 - p)^{\delta-m}\widehat{d}_m n \\ &\leq (1 - p)^\mu n \sum_{m=3}^{k-2} (k - m - 1)p\widehat{d}_m \\ &\leq e^{-p\mu}n \sum_{m=3}^{k-2} (k - m - 1)p\widehat{d}_m. \end{aligned}$$

For the next three terms, we obtain

$$\begin{aligned} kE(|B_0|) + (k - 1)E(|B_1|) + (k - 2)E(|B_2|) \\ \leq k(1 - p)^{\delta+1}n + (k - 1)p(1 - p)^\delta\widehat{d}_1n + (k - 2)p^2(1 - p)^{\delta-1}\widehat{d}_2n \\ \leq k(1 - p)^2e^{-p\mu}n + (k - 1)p(1 - p)d e^{-p\mu}n + (k - 2)p^2\widehat{d}_2e^{-p\mu}n. \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{m=3}^{k-1} (k - m)E(|B_m|) &\leq \sum_{m=3}^{k-1} (k - m)p^m(1 - p)^{\delta-m+1}\widehat{d}_m n \\ &\leq \sum_{m=3}^{k-2} (k - m)(1 - p)(1 - p)^{\delta-m}\widehat{d}_m n + (1 - p)^\mu\widehat{d}_{k-1}n \\ &\leq e^{-p\mu}n \left(\sum_{m=3}^{k-2} (k - m)(1 - p)\widehat{d}_m + \widehat{d}_{k-1} \right). \end{aligned}$$

Thus,

$$E(|D|) \leq pn + e^{-p\mu}n\Omega,$$

where

$$\begin{aligned} \Omega &= (k - 1)p(1 - p) + (k - 2)p^2d + (k - 3)p^3\widehat{d}_2 + \sum_{m=3}^{k-2} (k - m - 1)p\widehat{d}_m \\ &\quad + k(1 - p)^2 + (k - 1)p(1 - p)d + (k - 2)p^2\widehat{d}_2 + \sum_{m=3}^{k-2} (k - m)(1 - p)\widehat{d}_m + \widehat{d}_{k-1} \\ &\leq p^3(k - 3)\widehat{d}_2 + p^2(\widehat{d}_2(k - 2) - d + 1) + p(d(k - 1) - k - 1) + k + \sum_{m=3}^{k-1} (k - m)\widehat{d}_m. \end{aligned}$$

Taking into account that $k \geq 3$ and $\delta \geq 2$, we obtain

$$\begin{aligned} \widehat{d}_2(k-2) - d + 1 &\geq \widehat{d}_2 - d + 1 = \left(\sum_{i=1}^n 0.5d_i(d_i - 1) - \sum_{i=1}^n d_i + n \right) / n \\ &= \sum_{i=1}^n (0.5d_i^2 - 1.5d_i + 1) / n = \sum_{i=1}^n ((0.5d_i - 1)(d_i - 1)) / n \geq 0 \end{aligned}$$

and

$$d(k-1) - k - 1 = k(d-1) - d - 1 \geq 3(d-1) - d - 1 = 2d - 4 \geq 0.$$

Hence

$$\begin{aligned} \Omega &\leq (k-3)\widehat{d}_2 + \widehat{d}_2(k-2) - d + 1 + d(k-1) - k - 1 + k + \sum_{m=3}^{k-1} (k-m)\widehat{d}_m \\ &= (k-2)d + (2k-5)\widehat{d}_2 + \sum_{m=3}^{k-1} (k-m)\widehat{d}_m. \end{aligned}$$

If we denote the last expression by Ψ , then

$$E(|D|) \leq f(p) = pn + e^{-p\mu}n\Psi.$$

For $p \in [0, 1]$, the function $f(p)$ is minimised at the point $\min\{1, z\}$, where

$$z = \frac{\ln \mu + \ln \Psi}{\mu}.$$

If $z > 1$, then $f(p)$ is minimised at the point $p = 1$ and the result easily follows. If $z \leq 1$, then

$$E(|D|) \leq f(z) = \left(z + \frac{1}{\mu} \right) n = \frac{\ln \mu + \ln \Psi + 1}{\mu} n.$$

Since the expectation is an average value, there exists a particular k -tuple dominating set of order at most $f(z)$, as required. The proof of Theorem 4 is complete. \square

By a simple modification of the proof of Theorem 4, we obtain the following result:

Corollary 1. For any graph G with $3 \leq k \leq \delta + 1$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\sum_{m=1}^{k-1} (k-m)\widehat{d}_m - d) + 1}{\delta - k + 2} n.$$

Proof. If $k = 3$, then the result follows from Theorem 4. Thus, we may assume that $4 \leq k \leq \delta + 1$.

Using the notation of the proof of Theorem 4, we obtain

$$\begin{aligned} E(|D|) &\leq pn + (k-1)E(|A_0|) + (k-2)E(|A_1|) + \sum_{m=2}^{k-2} (k-m-1)E(|A_m|) \\ &\quad + kE(|B_0|) + (k-1)E(|B_1|) + \sum_{m=2}^{k-1} (k-m)E(|B_m|). \end{aligned}$$

Therefore,

$$E(|D|) \leq pn + e^{-p\mu}n\Omega,$$

where

$$\begin{aligned} \Omega &= (k-1)p(1-p) + (k-2)p^2d + \sum_{m=2}^{k-2} (k-m-1)p\widehat{d}_m \\ &\quad + k(1-p)^2 + (k-1)p(1-p)d + \sum_{m=2}^{k-2} (k-m)(1-p)\widehat{d}_m + \widehat{d}_{k-1} \\ &\leq p^2(1-d) + p(d(k-1) - k - 1) + k + \sum_{m=2}^{k-1} (k-m)\widehat{d}_m. \end{aligned}$$

If $k \geq 4$, then $d \geq \delta \geq 3$ and the function $p^2(1-d) + p(d(k-1) - k - 1)$ is monotonically increasing from 0 to 1. Therefore,

$$\Omega \leq \Psi = \sum_{m=1}^{k-1} (k-m)\widehat{d}_m - d.$$

The remaining part of the proof is similar to the final part of the proof of Theorem 4. \square

The next result summarizes all the above theorems and corollaries.

Corollary 2. For any graph G with $k \leq \delta + 1$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\sum_{m=1}^{k-1} (k-m)\widehat{d}_m + \varepsilon) + 1}{\delta - k + 2} n,$$

where $\varepsilon = 1$ if $k = 1$ or 2 , and $\varepsilon = -d$ if $k \geq 3$.

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